Possible Graduate Projects for Computational Physics

You may select another topic for a Graduate project, but the projects below will give you an idea of the depth expected in any project.

1) The Maximum Mass for a White Dwarf

The great astrophysicist, S. Chandrasekhar, did fundamental work on the nature of white dwarfs and, in particular, discovered that the maximum mass for a white dwarf is about 1.4 solar masses. Above that mass the white dwarf either undergoes a helium or carbon detonation (depending on the composition) or contracts to a neutron star.

Chandrasekhar's work led to the derivation (see "An Introduction to the Study of Stellar Structure, S. Chandrasekhar, Dover Publications 1939, 1967) of the differential equation for the internal structure of a white dwarf:

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + x(y^2 - C)^{3/2} = 0$$

where y is related to the gravitational potential in the interior of the star, x is the radial coordinate ($x = 0$ at the center of the star) and C is a mass parameter which may vary from 1 to 0. As $C \to 0$, the mass of the white dwarf increases while the radius of the white dwarf tends to 0. The point where the radius of the white dwarf is zero corresponds to a detonation or collapse into a neutron star. This sets the maximum mass for a white dwarf, called the Chandrasekhar Limit. Solutions for the above equation do not exist for $C \leq 0$. The density in the white dwarf, ρ, as a function of x is related to $y = y(x)$ in the following way:

$$\rho = \left(\frac{y^2 - C}{1 - C}\right)^{3/2}$$

The value of x for which ρ reaches 0 corresponds to the surface of the star and is denoted x_s. The total mass of the white dwarf is given by the equation:

$$M = -4\pi \left(\frac{2A}{\pi G}\right)^{3/2} \frac{1}{B^2} \left(x^2 \frac{dy}{dx}\right)_{x=x_s}$$

where M is in grams, and A, B, and G have the numerical values: $A = 6.002353 \times 10^{22}$, $B = 1.962044 \times 10^{6}$, and $G = 6.6742 \times 10^{-8}$. As pointed out above, as $C \to 0$, M approaches the maximum possible mass for a white dwarf. Since one cannot find a solution for the differential equation at $C = 0$, it will be necessary to compute M for various values of C approaching zero and then *extrapolate* to determine the value of M for $C = 0$.

You may need to use the following power series expansion for y near $x = 0$ to start the numerical integration of the differential equation:

$$y = 1 - \frac{q^3}{6}x^2 + \frac{q^4}{40}x^4 - \frac{q^5}{7!}(5q^2 + 14)x^6 + \ldots$$

where $q^2 = 1 - C$.

2) The Restricted Three Body Problem

It turns out that the graviational three body problem, in which three masses move under their mutual gravitational forces, cannot be solved in closed form. This is even true if the motion is restricted to a plane, and even if the three bodies consist of two massive bodies of masses M_1 and M_2 that revolve in circles about their common center of mass and a third very small (negligible in the sense that it does not affect the movements of M_1 and M_2) mass m that moves in the gravitational field of M_1 and M_2. We assume that m remains in the plane of motion of M_1 and M_2. This is called the *restricted three-body problem.*

If M_1 and M_2 are separated by a distance a, then the angular velocity, ω of their motion is given by

$$\omega^2 = \frac{(M_1 + M_2)G}{a^3}$$

We Introduce a coordinate system rotating with that angular velocity about the center of mass of M_1 and M_2. In that coordiate system, M_1 and M_2 are stationary. We take them to lie on the x-axis. Their coordinates on that axis are:

$$x_1 = \frac{M_2}{M_1 + M_2}a, \quad x_2 = \frac{M_1}{M_1 + M_2}a$$

where $\vec{\omega}$ is along the z-axis. The equation of motion of m in the xy-plane is given by:

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_1 + \vec{F}_2 - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \frac{d\vec{r}}{dt}$$

where $\vec{F}_1$ and $\vec{F}_2$ are the gravitational attraction of M_1 and M_2 on m. Written in terms of components, the two equations become

$$\ddot{x} = -\frac{M_1 G(x - x_1)}{[(x - x_1)^2 + y^2]^{3/2}} - \frac{M_2 G(x - x_2)}{[(x - x_2)^2 + y^2]^{3/2}} + \frac{(M_1 + M_2)Gx}{a^3} + 2\omega\dot{y}$$

$$\ddot{y} = -\frac{M_1 G y}{[(x - x_1)^2 + y^2]^{3/2}} - \frac{M_2 G y}{[(x - x_2)^2 + y^2]^{3/2}} + \frac{(M_1 + M_2)Gy}{a^3} - 2\omega\dot{x}$$

Notice m does not appear in these equations. These equations may be further simplified by "non-dimensionalizing" (left to the student). This can be accomplished by variable changes:

$$\xi = x/a, \quad \eta = y/a$$

$$\xi_1 = \frac{M_2}{M_1 + M_2}, \quad \xi_2 = -\frac{M_1}{M_1 + M_2} = \xi_1 - 1$$

With those substitutions the gravitational 'potential' in the rotating coordinate system, V, given by

$$V = -\frac{mM_1 G}{[(x - x_1)^2 + y^2]^{1/2}} - \frac{mM_2 G}{[(x - x_2)^2 + y^2]^{1/2}} - \frac{m(M_1 + M_2)G(x^2 + y^2)}{2a^3}$$

can be written:

$$V = \frac{m(M_1 + M_2)G}{a}\left\{\frac{\xi_2}{[(\xi - \xi_1)^2 + \eta^2]^{1/2}} - \frac{\xi_1}{[(\xi - \xi_2)^2 + \eta^2]^{1/2}} - \frac{1}{2}(\xi^2 + \eta^2)\right\}$$

The student can verify that the partials $\partial V/\partial\xi$ and $\partial V/\partial\eta$ *vanish* at 5 points (the Lagrangian points), two of them off the x-(ξ)axis (L_4 and L_5). An interesting computational problem is to release the test particle near either L_4 or L_5 with near-zero velocity in the rotating coordinate system, and compute the subsequent motion. Since L_4 and L_5 are semi-stable points for $M_2 < 0.04M_1$, the particle should execute a complex orbit about those points. For more information, see Mechanics 3rd edition, by Symon, 1971 or any junior-level or graduate-level text in Classical Mechanics.

3) The Problem of Saturn's Rings

Saturn's rings are planar and exhibit a number of gaps, the most prominent of which is Cassini's division. The rings are made up of large numbers of co-orbiting particles, some as small as a grain of dust, others a few meters across. Immediately outside the rings orbit a number of small moons. It has been known for many years that the orbital period of Mimas is exactly twice that of a particle in Cassini's division, and it is clear that the gravitational force of Mimas on such a particle increases its eccentricity and thus knocks it out of the gap. Thus, Mimas has cleared Cassini's division via a gravitational resonance. This situation is exactly the restricted three-body problem in (2) above with Saturn as M_1 and Mimas as M_2. It is possible to simulate the clearing of Cassini's division by introducing a large number (1000) of "test particles" in circular orbits between Saturn and the orbit of Mimas and tracing their orbits with time.

4) The Partition Function of Hydrogen

The hydrogen atom has essentially an infinite number of bound energy levels whose energies are given by the equation:

$$E_n = -R\left(\frac{1}{n^2} - 1\right)$$

where R is the Rydberg constant with value 109677.58311 cm^{-1} (for hydrogen), and n is the principle quantum number of the energy level. The partition function is given by

$$U(T) = \sum g_i \exp(-E_i/kT)$$

where g_i is the statistical weight of the i^{th} level, such that $g_n = 2n^2$, k is the Boltzmann constant (use the value appropriate for the energy units used for E_i) and T is the temperature in kelvins. The summation is over all *bound* states, which, in a vacuum, is infinite, and converge on the ionization energy $E_{\text{ion}} = R$. However, in a plasma, the electric fields of neighboring atoms and electrons strip some of the upper energy levels off effectively reducing the ionization energy by a quantity ΔE. This means that the last energy level in the summation is the one with energy just below $E_{\text{ion}} - \Delta E$. Write a program that will prompt the user for ΔE, calculate $U(T)$ for temperatures between 1000 and 120,000 K, and plot it with gnuplot.

5) Chaos in the Lorentz Model

It used to be believed that Newtonian mechanics was strictly deterministic, that is, give the position of a particle in space, and knowledge of all the forces on it, it would be possible to predict its position at all later times. However, in the early 1960's, an MIT meteorologist Ed Lorenz was investigating the governing equations of weather and discovered a system of equations that, while derived from Newtonian mechanics, were not deterministic, as they displayed a property now known as "chaos" in that the solution was extremely sensitive to the initial conditions. That system of equations is given by:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = rx - y - xz$$

$$\frac{dz}{dt} = xy - bz$$

Write a program that will integrate these three differential equations simultaneously, and then figure out how to display the solution in three dimensions in gnuplot. Use $\sigma = 10$, $b = 8/3$ and $r = 28$. Use the initial conditions $x = 1$, $y = 1$, and $z = 20.01$. Experiment with the initial conditions to see how the system behaves. Try different values for σ, b, and r to see if the system is chaotic over the entire parameter space.

6) A project of your choosing

If you have a computational project of similar "difficulty" that you are interested in, come to my office and describe it for me. If it is acceptable, I will need from you early in the semester a short write-up (similar to those above) describing the problem and what you will do to solve it.