A NOTE ON COMPACTNESS OF COUNTABLE SETS

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Abstract. The Heine/Borel compactness theorem restricted to subsets of the rationals implies $\text{WKL}_0$.

One of the earliest results in reverse mathematics was Friedman’s theorem on the equivalence of $\text{WKL}_0$ and the Heine/Borel theorem for the unit interval $[0,1]$. In June 1999, he asked if the restriction of the Heine/Borel theorem to countable closed subsets implied $\text{WKL}_0$. The following theorem provides an affirmative answer, even in the case when the countable subsets consist entirely of rational numbers.

Theorem 1 ($\text{RCA}_0$). The following are equivalent:
1. $\text{WKL}_0$
2. Every closed set of rationals in $[0,1]$ is Heine/Borel compact.

Proof. Friedman [2] proved that $\text{WKL}_0$ implies the Heine/Borel theorem for $[0,1]$. (See [3] for the proof.) For the result above, only the reversal requires proof. Emulating Simpson’s reversal of the full Heine/Borel theorem ([3] pages 128–129), for each $s \in 2^{<\aleph_0}$, let

$$a_s = \sum_{i < \text{lth}(s)} \frac{2^i s(i)}{3^i + 1} \quad \text{and} \quad b_s = a_s + \frac{1}{3^{\text{lth}(s)}}.$$

Also, let $c_s = \frac{1}{2}(a_s + b_s)$ and define the intervals

$$M_s = (c_s - \frac{1}{2 \cdot 3^{\text{lth}(s)+1}}, c_s + \frac{1}{2 \cdot 3^{\text{lth}(s)+1}}).$$

Note that for every $s$, the interval $[a_s, b_s]$ is divided into a left third $[a_s^{-}(0), b_s^{-}(0)]$, a middle third $M_s$ centered at $c_s$, and a right third $[a_s^{+}(1), b_s^{+}(1)]$. Since $[a_s^{+}(1), b_s^{+}(1)] = [0,1]$, these intervals correspond to those in the usual Cantor construction. As in [3], let

$$a_s' = a_s - \frac{1}{3^{\text{lth}(s)+1}} \quad \text{and} \quad b_s' = b_s + \frac{1}{3^{\text{lth}(s)+1}}.$$

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The interval \((a', b')\) is enough wider than \((a, b)\) that it includes the points \(\{c_t \mid t \geq s\}\) and every limit point of this set. Furthermore, if \(t \not\geq s\), then \(c_t \notin (a', b')\).

Let \(T\) be a 0–1 tree with no infinite paths. We will prove the desired reversal by showing that 2) implies that \(T\) is finite. Let \(C = \{c_s \mid s \in T\}\). We must show that \(C\) is closed. Let \(U\) be the open set consisting of the union of

\[\{M_s - \{c_s\} \mid s \in 2^{\mathbb{N}}\}\]

and

\[\{(a_{s\ominus}(\epsilon), b'_{s\ominus}(\epsilon)) \mid s \in T \land \epsilon \in \{0, 1\} \land s\ominus(\epsilon) \notin T\}.
\]

We can show that \(C\) is the complement of \(U\) by showing that for each \(x \in [0, 1]\), \(x\) is in either \(C\) or \(U\), but not both. First suppose that \(x\) is not in the Cantor middle third set. Then for some \(s\), \(x \in M_s\). Thus \(x = c_s \in C\) or \(x \in M_s - c_s \subset U\). Since \(M_s \cap C = \{c_s\}\), only one of these possibilities can hold. Now suppose that \(x\) is in the Cantor middle third set. Then for some \(f : \mathbb{N} \to \{0, 1\}\), we have

\[x = \sum_{i=0}^{\infty} \frac{2 \cdot f(i)}{3^{i+1}} \notin C.
\]

Since \(T\) contains no infinite paths, there is an \(m \in \mathbb{N}\) such that \(f[m] \in T\) but \(f[m+1] \notin T\). Thus \(x \in (a'_{f[m+1]}, b'_{f[m+1]}) \subseteq U\), completing the proof that \(C\) is the complement of \(U\).

We have shown that \(C\) is a closed set of rationals in \([0, 1]\). Consider the open cover of \(C\) given by \(V = \{M_s \mid s \in T\}\). By 2), \(C\) is covered by a finite subset of \(V\). Each element of \(V\) contains exactly one element of \(C\), so \(C\) is finite. Each sequence in \(T\) corresponds to exactly one element of \(C\), so \(T\) is finite.

In the preceding theorem, a closed set was viewed as the complement of an open set. We say that \(X\) is separably closed if there is a countable set of points \(X\) such that \(\bar{X}\) is the set of limit points of \(X\). If the closed sets of the preceding result are replaced by separably closed sets, the logical strength of the Heine/Borel theorem changes, yielding the following equivalence.

**Theorem 2 (RCA\(_0\)).** The following are equivalent:

1. ACA\(_0\)
2. Every separably closed set of rationals in \([0, 1]\) is Heine/Borel compact.

**Proof.** To prove that ACA\(_0\) proves 2), note that ACA\(_0\) proves that every separably closed set is closed [1]. To prove the reversal, let \(f : \mathbb{N} \to \mathbb{N}\) be an injection, and define \(S = \langle r_i \mid i \in \mathbb{N}\rangle\) by

\[r_i = \sum_{j=0}^{i} 2^{-f(j)-1}.
\]

Consider the set of open intervals \(\{(0, r_i) \mid i \in \mathbb{N}\}\). Any element of \(\bar{S}\) which is irrational or not contained in the intervals encodes the range of \(f\).
REFERENCES


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