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Hilbert versus Hindman

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Draft version

Abstract We show that a statement HIL, which is motivated by a lemma of Hilbert and close in formulation to Hindman's theorem, is actually much weaker than Hindman's theorem. In particular, HIL is finitistically reducible in the sense of Hilbert's program, while Hindman's theorem is not.

Keywords Reverse mathematics · pigeonhole · conservation · finitism

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Brown, Erdős, Chung, and Graham [2] point out that a finitary lemma of Hilbert [3] can be viewed as a version of Folkman's theorem with a relaxed condition on the monochromatic set. By analogy, in the infinite setting there is a remarkable similarity between a certain version of Hindman's theorem and the following statement, which the author heard about from Henry Towsner [10].

HIL: Suppose $f : \mathbb{N}^{<\mathbb{N}} \to k$ is a finite coloring of the finite subsets of the natural numbers. Then there is a an infinite sequence $\langle X_i \rangle_{i\in\mathbb{N}}$ of distinct finite sets and a color $c < k$ such that for every finite set $F \subset \mathbb{N}$ we have $f(\cup_{i\in F} X_i) = c$.

Note that the finite union form of Hindman's theorem, denoted by HTU by Blass, Hirst, and Simpson [1], differs from HIL only in requiring that the sequence of sets is pairwise disjoint rather than merely distinct. This small change in wording has disproportionate impact on the strength of the theorem and the complexity of its proof.

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We use the techniques of reverse mathematics to compare the strengths of the two theorems. Any axiom systems used below are described in detail in Simpson's book [8]. Blass, Hirst, and Simpson [1] show that over RCA_0, Hindman's theorem implies ACA_0. Technically, their version of Hindman's theorem requires that $\max(X_i) < \min(X_{i+1})$ for each i, but that version is equivalent to the pairwise disjoint version over RCA_0. By contrast, HIL is equivalent to the much weaker infinite pigeonhole principle, often denoted by $\mathsf{RT}(1)$.

Theorem 1 (RCA_0) *The following are equivalent:*

1. HIL.
2. $\mathsf{RT}(1)$*: If* $f : \mathbb{N} \to k$ *then there is a* $c < k$ *such that* $\{n \mid f(n) = c\}$ *is infinite.*

Proof We work throughout in RCA_0. To show that HIL implies $\mathsf{RT}(1)$, assume HIL and let $f : \mathbb{N} \to k$ be a finite coloring of $\mathbb{N}$. By Δ^0_1 comprehension, there is a coloring $g : \mathbb{N}^{<\mathbb{N}} \to k$ defined by $g(X) = f(\max(X))$. Apply HIL to g, and thin the resulting sequence so that $i < j$ implies $\max(X_i) < \max(X_j)$. Then for every $x \in \{\max(X_i) \mid i \in \mathbb{N}\}$ we have $f(x) = g(X_0)$.

To prove the converse, assume $\mathsf{RT}(1)$ and let $f : \mathbb{N}^{<\mathbb{N}} \to k$ be a finite coloring of $\mathbb{N}^{<\mathbb{N}}$. Let $[0, n]$ denote the set $\{0, 1, 2, \ldots, n\}$. By Δ^0_1 comprehension, there is a coloring $g : \mathbb{N} \to k$ defined by $g(n) = f([0, n])$. Apply $\mathsf{RT}(1)$ to g to find an infinite sequence $n_0 < n_1 < n_2 < \ldots$ and a color $c < k$ such that $g(n_i) = c$ for all i. The intervals in the sequence $[0, n_0], [0, n_1], \ldots$ are distinct, and for each i we have $f([0, n_i]) = c$. Since for every finite set F, the union $\cup_{i\in F}[0, n_i]$ is equal to the interval $[0, n_{\max(F)}]$, the sequence witnesses that HIL holds.

From a naïve comparison of the preceding argument to any proof of Hindman's theorem, Hindman's theorem is much more difficult to prove than HIL. From a computability theoretic viewpoint, the previous theorem shows that HIL is computably true, but Hindman's theorem is known to imply the existence of the jump [1]. The reverse mathematical analysis also allows us to distinguish between the results through the lens of finitistic reductionism.

Based on work of Tait [9], Simpson [7] argues that Hilbert's program can be partially realized by determining that portion of mathematics which is finitistically reducible, that is, provable in axiom systems that are conservative over PRA (primitive recursive arithmetic) for Π^0_1 formulas. The following well-known result implies that $\mathsf{RCA}_0 + \mathsf{HIL}$ satisfies this criterion.

Theorem 2 $\mathsf{WKL}_0 + \mathsf{RT}(1)$ *(and hence* $\mathsf{RCA}_0 + \mathsf{HIL}$*) is conservative over* PRA *for* Π^0_2 *formulas.*

Proof The proof for $\mathsf{WKL}_0 + \mathsf{RT}(1)$ is a minor modification of a proof of Friedman's conservation result for WKL_0, presented as Theorem IX.3.16 by Simpson [8]. Based on Harrington's primitive recursive indicator for regular initial segments of nonstandard models of fragments of arithmetic, Paris [5] notes that the semi-regular initial segments are symbiotic with the regular initial segments. This means that given any semi-regular cut I selected as in Simpson's Lemma IX.3.13 with $b \in I < c$, there is a regular cut J with $b \in J < c$. Substituting J in the proof of Theorem IX.3.16 yields a model of $\mathsf{WKL}_0 + \mathsf{B}\Sigma^0_2$, which is equivalent to $\mathsf{WKL}_0 + \mathsf{RT}(1)$ by a result of Hirst [4]. Since WKL_0 includes RCA_0, by Theorem 1 the result for $\mathsf{RCA}_0 + \mathsf{HIL}$ follows.

While HIL is finitistically reducible in the sense of Hilbert's program, Hindman's theorem is not. As a consequence of Proposition 1.6 and Proposition 3.1 of Sieg [6], ACA_0 proves the consistency of PRA. This statement can be formalized as a Π^0_1 statement and, by Gödel's second incompleteness theorem, is not provable in PRA. Since Hindman's theorem implies ACA_0 [1], Hindman's theorem is not conservative over primitive recursive arithmetic for Π^0_1 formulas, and so is not finitistically reducible. Summarizing, HIL is amenable to Hilbert's program, but Hindman's theorem exceeds the bounds of finitistic reductionism.

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