Disguising induction: Proofs of the pigeonhole principle for trees

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ABSTRACT. We examine the relationship between a pigeonhole principle for trees and induction on Σ²_0 formulas. This analysis is carried out in the framework of reverse mathematics utilizing a hierarchy of axiom systems formulated by Harvey Friedman.

Let \(2^N\) denote the set of all finite sequences of zeros and ones. We often use \(\sigma\) to denote both a finite sequence and the associated finite function, so \(\sigma(0)\) is the first element of the sequence, and \(\sigma(lh(\sigma) - 1)\) is the last. If \(\tau\) consists of \(\sigma\) with appended elements we write \(\sigma \subseteq \tau\), and write \(\sigma \subset \tau\) when \(\tau\) is a proper extension of \(\sigma\). Viewing \(2^<N\) as a partial order ordered by the \(\subseteq\) relation, we can think of any subset of \(2^<N\) as a subtree. A bijection between a subset \(S \subseteq 2^<N\) and \(2^<N\) that preserves extension is an order isomorphism. Using this terminology, we can formulate the following pigeonhole principle on binary trees.

**TT(1):** Suppose \(f : 2^<N \to n\) for some \(n \in \mathbb{N}\). Then there is a subtree \(S \subseteq 2^<N\) order isomorphic to \(2^<N\) and a \(c < n\) such that \(f(\sigma) = c\) for every \(\sigma \in S\).

This pigeonhole principle follows immediately from a version of Hindman’s theorem. If we let \(\text{FIN}\) denote the collection of all nonempty finite subsets of \(\mathbb{N}\), then the familiar finite sum form of Hindman’s theorem [9] is equivalent to the following statement. (See [1].)

**HT:** Suppose \(f : \text{FIN} \to n\) for some \(n \in \mathbb{N}\). Then there is a sequence \(\langle X_i \rangle_{i \in \mathbb{N}}\) of elements of \(\text{FIN}\) and a \(c < n\) such that

- if \(i < j\) then \(\max(X_i) < \min(X_j)\), and

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for every finite nonempty set $J \subset \mathbb{N}$, $f(\bigcup_{j \in J} X_j) = c$.

Here is a proof that HT implies TT(1). Suppose $f : 2^{<\mathbb{N}} \to n$. For each $X \in \text{FIN}$, let $\sigma_X$ be a sequence in $2^{<\mathbb{N}}$ of length $\max(X) + 1$ such that for each $i$, $\sigma_X(i) = 1$ if and only if $i \in X$. Define $g : \text{FIN} \to n$ by $g(X) = f(\sigma_X)$. Apply HT to $g$, and let $c$ be the associated color. Define $Y_{(j)} = \emptyset$, and for each nonempty $\tau \in 2^{<\mathbb{N}}$, let $Y_\tau$ be the union of $X_0$ or $X_1$, $X_2$ or $X_3$, $X_4$ or $X_5$ and so on, where the set $X_{2i}$ is included if $\tau(i) = 0$ and $X_{2i+1}$ is included if $\tau(i) = 1$. More formally, for nonempty $\tau \in 2^{<\mathbb{N}}$, let $Y_\tau = \bigcup_{j \in \text{FIN}(\tau)} X_{2i+1}$. Then the set $S = \{ \sigma_{Y_\tau} \mid \tau \in 2^{<\mathbb{N}} \}$ is a subtree of $2^{<\mathbb{N}}$ and the map taking $\tau$ to $\sigma_{Y_\tau}$ is an order isomorphism between $2^{<\mathbb{N}}$ and $S$. Furthermore, for each $\tau$, $f(\sigma_{Y_\tau}) = g(Y_\tau) = c$, so $S$ is the desired monochromatic subtree.

Timothy McNicholl [11] asked if the use of Hindman’s theorem is actually necessary to prove TT(1). Reverse mathematics, based on the axiom systems formulated by Harvey Friedman [6, 7], provides an excellent tool set for addressing this type of question. Indeed, HT is not needed to prove TT(1), as was shown in [3]. We will present a proof of this result below.

We will formalize our proof in the subsystem RCA$_0$. This theory has variable types for natural numbers and sets of natural numbers, basic arithmetic axioms including induction restricted to $\Sigma^0_1$ formulas, and the recursive comprehension axiom, which (naively) asserts the existence of computable sets. A very detailed discussion of RCA$_0$ can be found in Simpson’s book [12]. Since TT(1) implies the infinite pigeonhole principle, it cannot be proved in RCA$_0$ [10]. However, if we append the induction scheme for $\Sigma^0_2$ formulas, denoted by $\Sigma^0_2 - \text{IND}$, then TT(1) can be proved, as stated in the following result.

**THEOREM 1.** (RCA$_0$) $\Sigma^0_2 - \text{IND}$ implies TT(1).

**Proof.** We carry out the proof in RCA$_0$. Suppose $f : 2^{<\mathbb{N}} \to n$. Let $\{X_j \mid j < 2^n\}$ be the collection of all subsets of $\{0, 1, \ldots, n-1\}$, enumerated so that $X_i \subseteq X_j$ implies $i \leq j$. Since the entire range of $f$ is included in some $X_j$, there is a $j$ such that

$$\exists \sigma \forall \tau \supseteq \sigma \ (f(\tau) \in X_j),$$

which is clearly a $\Sigma^0_2$ formula when the finite sequences are identified with their natural number codes. $\Sigma^0_2 - \text{IND}$ implies that there is a least such $j$. (For equivalent formulations of induction schema, see Theorem 2.4 of [8].) Call this least element $j_0$, and choose $\sigma_0$ such that $\forall \tau \supseteq \sigma_0 \ (f(\tau) \in X_{j_0})$. If $c = f(\sigma_0)$, then every $\tau \supseteq \sigma_0$ can be extended to an element $\tau'$ with $f(\tau') = c$. We can construct a monochromatic subtree order isomorphic to
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2<^N by the following process. Let \( \sigma_0 \) be the root node. Fix a level-by-level enumeration of 2<^N. If \( \sigma \) has been added to the tree, for each \( i \in \{0, 1\} \) let \( \tau_i \) be the first enumerated extension of \( \sigma \updownarrow i \) that has color \( c \). Note that recursive comprehension suffices to prove the existence of this subtree. ■

Imitating [3], we will call the monochromatic tree of the preceding proof the standard tree of color \( c \) based at \( \sigma_0 \). The fact that \( HT \) is not required for the proof of \( TT(1) \) is now an easy corollary.

**COROLLARY 2.** RCA\(_0\) + TT(1) does not prove HT.

**Proof.** The natural numbers together with the computable sets form a model of RCA\(_0\) + \( \Sigma^0_2 - \text{IND} \), but not a model of arithmetical comprehension (since not all arithmetically definable sets are computable.) This model is not a model of HT, since RCA\(_0\) + HT implies arithmetical comprehension (Theorem 2.6 of [1]). ■

Although this shows that HT is not necessary to prove TT(1), it leads us to ask if \( \Sigma^0_2 - \text{IND} \) is necessary to prove TT(1) and exploring their relationships to \( \Sigma^0_2 - \text{IND} \).

### 1 Eventually constant tails

The core of the proof of Theorem 1 is locating a node \( \sigma \) such that for every extension \( \alpha \) of \( \sigma \) the spectrum of colors appearing above \( \alpha \) is exactly the same as the spectrum of colors appearing above \( \sigma \). We can formalize the existence of such a \( \sigma \) as follows:

**ECT(2<^N):** If \( f : 2<^N \to n \) then \( \exists \sigma \forall \tau \supseteq \sigma \forall \alpha \supseteq \sigma \exists \beta \supset \alpha (f(\tau) = f(\beta)). \)

This same principle can be expressed for colorings of \( \mathbb{N} \).

**ECT(\mathbb{N}):** If \( f : \mathbb{N} \to n \) then \( \exists b \forall x \geq b \exists y > x (f(x) = f(y)). \)

Informally, ECT(\mathbb{N}) asserts that there is a \( b \) such that whenever \( [x, \infty) \subseteq [b, \infty) \) then the range of \( f \) restricted to \( [x, \infty) \) is identical to the range of \( f \) restricted to \( [b, \infty) \). That is, we are saying that the range of \( f \) on tails is eventually constant. The label ECT is a mnemonic for eventually constant tails.

ECT(2<^N) can be substituted for the use of \( \Sigma^0_2 - \text{IND} \) in the proof of Theorem 1. The next three lemmas explore the relationships between the two forms of ECT and \( \Sigma^0_2 - \text{IND} \), and the consequences are collected in Theorem 6.

**LEMMA 3.** (RCA\(_0\)) ECT(2<^N) implies ECT(\mathbb{N}).
Proof. Working in RCA₀, fix \( f : \mathbb{N} \to n \). Define \( g : 2^{<\mathbb{N}} \to n \) by \( g(\sigma) = f(\text{lh}(\sigma)) \). Apply ECT\((2^{<\mathbb{N}})\) to \( g \) and obtain \( \sigma \). Let \( b = \text{lh}(\sigma) \) and choose \( x \geq b \). Choose any \( \tau \supseteq \sigma \) with \( \text{lh}(\tau) = x \). By ECT\((2^{<\mathbb{N}})\), there is a \( \beta \supseteq \tau \) such that \( g(\beta) = g(\tau) \). Thus \( \text{lh}(\beta) > x \) and \( f(\text{lh}(\beta)) = g(\beta) = g(\tau) = f(\tau) \). □

LEMMA 4. \((\text{RCA}_0)\) ECT\((\mathbb{N})\) implies \( \Sigma^0_4 - \text{IND}\).

Proof. By Exercise 11.3.13 of [12], over \(\text{RCA}_0\) the scheme \( \Sigma^0_4 - \text{IND} \) is equivalent to the bounded \( \Sigma^0_4 \) comprehension scheme, which is the assertion that

\[
\forall n \exists X \forall i (i \in X \leftrightarrow (i < n \land \varphi(i)))
\]

where \( \varphi(i) \) is a \( \Sigma^0_4 \) formula not containing \( X \). We will use ECT\((\mathbb{N})\) to derive bounded \( \Sigma^0_4 \) comprehension.

Suppose \( \varphi(i) \) is \( \exists x \forall y \theta(i, x, y) \), where \( \theta \) is quantifier free. Fix \( n \). Define \( f : \mathbb{N} \to n + 1 \) by writing each natural number as \( mn + i \) with \( i < n \) and setting

\[
f(mn + i) = \begin{cases} i & \text{if } \mu x < m \forall y < m \theta(i, x, y) \text{ is not equal to } \mu x < m + 1 \forall y < m + 1 \theta(i, x, y) \\ n & \text{otherwise.} \end{cases}
\]

In the preceding, when \( \forall x < m \exists y < m \neg \theta(i, x, y) \), we define the expression \( \mu x < m \forall y < m \theta(i, x, y) \) to be equal to \( m \).

Note that if there is an \( x \) such that \( \forall y \theta(i, x, y) \) then \( \Sigma^0_4 - \text{IND} \) implies there is a least such \( x \); call it \( x_0 \). Applying \( \text{BSX}_0^0 \), which is also a consequence of \( \Sigma^0_1 - \text{IND} \) (see [8], Chapter I, section 2), we can find an \( m_0 \) so large that \( \forall t < x_0 \neg \forall y < m_0 \theta(i, t, y) \). Then for any \( m > m_0 \),

\[
\mu x < m \forall y < m \theta(i, x, y) = x_0 = \mu x < m + 1 \forall y < m + 1 \theta(i, x, y),
\]

so \( f(mn + i) = n \). Consequently, on any final segment of \([m_0 n + i, \infty)\), \( i \) is not in the range of \( f \). Summarizing, if \( \exists x \forall y \theta(i, x, y) \), then eventually \( i \) is omitted from all tails of \( f \).

On the other hand, suppose \( \neg \exists x \forall y \theta(i, x, y) \) and fix an element \( b > 0 \). If for every \( m > b \) we have \( \forall x < m \exists y < m \neg \theta(i, x, y) \), then for all \( m > b \) we have \( f(mn + i) = i \) and so eventually \( i \) is in every tail. Suppose that for some \( m > b \) we have \( \exists x < m \forall y < m \theta(i, x, y) \). Let \( x_0 \) be the least element less than \( m \) satisfying \( \forall y < m \theta(i, x_0, y) \). Since \( \neg \exists x \forall y \theta(i, x, y) \), let \( y_0 \) be the least element such that \( \neg \theta(i, x_0, y_0) \). Then \( \mu x < y_0 \forall y < y_0 \theta(i, x, y) = x_0 \), but \( \mu x < y_0 + 1 \forall y < y_0 + 1 \theta(i, x, y) > x_0 \). Hence \( f(y_0 n + i) = i \) for arbitrarily large values of \( n \). Summarizing, if \( \neg \exists x \forall y \theta(i, x, y) \) then eventually \( i \) is in every tail.
Apply ECT(\(\mathbb{N}\)) to \(f\) to find a \(b\) such that the range of \(f\) is constant on final segments of \([b, \infty)\). By bounded \(\Sigma^0_1\) comprehension (which is a consequence of \(\Sigma^0_1 - \text{IND}\)) the set

\[Y = \{i < n \mid \exists t > b \ f(t) = i\}\]

exists. By recursive comprehension, the set \(X = \{i < n \mid i \notin Y\}\) also exists, and by the preceding paragraphs, \(i \in X\) if and only if \(i < n\) and \(\exists x \forall y \theta(i, x, y)\), as desired. 

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LEMMA 5. \((\text{RCA}_0)\) \(\Sigma^0_2 - \text{IND}\) implies ECT(\(2^{<\mathbb{N}}\)).

**Proof.** Suppose \(f : 2^{<\mathbb{N}} \to \mathbb{N}\). Let \(\langle X_i \mid i < 2^n \rangle\) enumerate the subsets of \(\mathbb{N}\) in an order that preserves containment. By the \(\Sigma^0_2\) least element principle (which is equivalent over \(\text{RCA}_0\) to \(\Sigma^0_2 - \text{IND}\)) \([8]\), there is a least \(j\) such that we can find a node \(\sigma\) so that \(\forall \tau \supseteq \sigma (f(\tau) \in X_j)\). Since \(j\) is the least such integer, for every \(\tau \supseteq \sigma\), the spectrum of colors appearing above \(\tau\) must match that above \(\sigma\). 

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THEOREM 6. \((\text{RCA}_0)\) The following are equivalent:

1. ECT(\(2^{<\mathbb{N}}\)).
2. ECT(\(\mathbb{N}\)).
3. \(\Sigma^0_2 - \text{IND}\).

**Proof.** Immediate from Lemmas 3, 4, and 5. 

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Thus, the ECT principles are essentially disguised forms of \(\Sigma^0_2\) induction. This is a common attribute among many principles that imply \(\mathbb{T}(1)\), as shown in the next section.

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2 A result of Corduan, Groszek, and Mileti

Doctors Corduan, Groszek, and Mileti reveal a strong connection between \(\mathbb{T}(1)\) and \(\Sigma^0_2 - \text{IND}\) in the following conservation result, which appears in \([4]\).

THEOREM 7. If \(\Sigma\) is any extension of \(\text{RCA}_0\) by \(\Pi^1_1\) axioms, then \(\Sigma\) proves \(\mathbb{T}(1)\) if and only if \(\Sigma\) proves \(\Sigma^0_2 - \text{IND}\).

Since the usual infinite pigeonhole principle, \(\text{RT}(1)\), can be expressed as a \(\Pi^1_1\) formula and is known to be equivalent to the scheme \(\mathbb{B}\Pi^1_1\) and therefore strictly weaker than \(\Sigma^0_2 - \text{IND}\), an immediate corollary of Theorem 7 is that \(\text{RT}(1)\) does not imply \(\mathbb{T}(1)\) over \(\text{RCA}_0\). This result appears as Corollary...
3.8 in [4] and provides the current best strict lower bound on the strength of \( TT(1) \).

Using Theorem 7, we can show that many informal proofs of \( TT(1) \) make use of disguised forms of \( \Sigma^0_2 - \text{IND} \). For example, in the next paragraph we present an alternative proof of a portion of Theorem 6. This new argument sidesteps the technical details of Lemma 4, but also does not yield information about \( ECT(N) \).

**Proof.** [Alternative proof that \( ECT(2^{<N}) \) implies \( \Sigma^0_2 - \text{IND} \) over \( RCA_0 \).] By Theorem 7, since \( ECT(2^{<N}) \) is a \( \Pi^1_1 \) sentence it suffices to prove that \( ECT(2^{<N}) \) implies \( TT(1) \) over \( RCA_0 \). Working in \( RCA_0 \), suppose \( f : 2^{<N} \to n \) and apply \( ECT(2^{<N}) \) to find a \( \sigma \) such that every extension of \( \sigma \) can be further extended to a node \( \beta \) with \( f(\beta) = f(\sigma) \). As in the proof of Theorem 1, \( RCA_0 \) proves the existence of the standard tree of color \( f(\sigma) \) based at \( \sigma \). ■

The following limit principle arises as a natural intermediate step in a proof of \( TT(1) \) from stable Ramsey’s Theorem for pairs (denoted \( SRT^2 \)).

**L:** Given \( f : N^2 \to a \) such that \( \lim_n f(x,n) \) exists for every \( x \in N \), there is a least \( b < a \) such that \( \exists \sigma \exists t \forall n (n > t \to f(x,n) = b) \). By the \( \Sigma^0_2 \) least element principle, which is equivalent to \( \Sigma^0_2 - \text{IND} \), there is a least such \( b \). Thus \( L^+ \) holds. Since predicate calculus proves that \( L^+ \) implies \( L \), the last sentence of the lemma follows immediately. ■

By proving that \( L \) implies \( TT(1) \), we create an opportunity for applying Theorem 7.

**Lemma 9.** \( (RCA_0) \ L \ implies \ TT(1) \).

**Proof.** Let \( f : 2^{<N} \to a \) and let \( \langle \sigma_i \rangle_{i \in N} \) be an enumeration of \( 2^{<N} \). Let \( \langle Y_i \rangle_{i < 2^a} \) be an enumeration of the power set of \( \{0,1,\ldots,a-1\} \) such that \( Y_i \subseteq Y_j \) implies \( i \leq j \). Define \( g : N^2 \to 2^a \) by

\[
g(m,n) = \mu(\{f(\tau) \mid \tau \supseteq \sigma_m \land \text{lh}(\tau) \leq n\} = Y_i).
\]
Note that recursive comprehension suffices to prove the existence of \( g \) and that for fixed \( m \), the function \( g(m, n) \) is increasing. By the \( \Pi_1^1 \) least element principle (which is provable in \( \mathsf{RCA}_0 \)), for each \( m \) there is a least upper bound on the range of \( g(m, n) \). Consequently, for each \( m \), \( \lim_n g(m, n) \) exists. By the \( \Pi_0^1 \) least element principle (which is provable in \( \mathsf{RCA}_0 \)), for each \( m \) there is a least upper bound on the range of \( g(m, n) \). Consequently, for each \( m \), \( \lim_n g(m, n) = b \) also. In particular, every node extending \( \sigma_m \) can be extended to a node of color \( f(\sigma_m) \). Thus, the standard tree of color \( f(\sigma_m) \) extending \( \sigma_m \) (as constructed in the proof of Theorem 1) is isomorphic to \( 2^{<\mathbb{N}} \).

Combining Theorem 7 with the preceding lemmas, we see that \( \mathsf{L} \) and \( \mathsf{L}^+ \) are disguised forms of \( \Sigma_0^2 - \mathsf{IND} \).

**Corollary 10.** (\( \mathsf{RCA}_0 \)) The following are equivalent:

1. \( \Sigma_0^2 - \mathsf{IND} \).
2. \( \mathsf{L}^+ \).
3. \( \mathsf{L} \).

**Proof.** Lemma 8 shows that over \( \mathsf{RCA}_0 \), item 1 implies item 2, and item 2 implies item 3. Since \( \mathsf{L} \) is \( \Pi_1^1 \), Theorem 7 applied to Lemma 9 shows that \( \mathsf{RCA}_0 \) proves that item 3 implies item 1.  

We close the section by noting that any proof of \( \mathsf{TT}(1) \) relying on the standard tree construction from the proof of Theorem 1 inherently uses \( \Sigma_0^2 - \mathsf{IND} \).

**Theorem 11.** (\( \mathsf{RCA}_0 \)) The following are equivalent:

1. \( \Sigma_0^2 - \mathsf{IND} \).
2. If \( f : 2^{<\mathbb{N}} \rightarrow a \) then there is a node \( \sigma \) such that the standard tree of color \( f(\sigma) \) based at \( \sigma \) is isomorphic to \( 2^{<\mathbb{N}} \).

**Proof.** The proof of Theorem 1 shows that item 1 implies item 2. To prove the reversal, note that over \( \mathsf{RCA}_0 \), item 2 is equivalent to the statement “if \( f : 2^{<\mathbb{N}} \rightarrow a \) then there is a node \( \sigma \) such that for every \( n \) there is a stage \( t \) such that the algorithm for constructing the standard tree of color \( f(\sigma) \) based at \( \sigma \) halts and produces a tree containing an initial segment order isomorphic to the full binary tree of height \( n \).” Since this statement is \( \Pi_1^1 \) and implies \( \mathsf{TT}(1) \) over \( \mathsf{RCA}_0 \), by Theorem 7 it implies \( \Sigma_0^1 - \mathsf{IND} \). Thus, item 2 implies item 1 over \( \mathsf{RCA}_0 \).
3 Null stable colorings

In this section, we present one more disguised form of induction, a related though weaker combinatorial principle that is actually equivalent to TT(1), and a proof of TT(1) from a stable version of Ramsey’s theorem for pairs in trees. All these results depend on the following notion. Suppose we have \( f : 2^{<\mathbb{N}} \to 2 \). We say that \( f \) is a null stable coloring if for every \( \sigma \in 2^{<\mathbb{N}} \) there is a \( \sigma' \supseteq \sigma \) such that for every \( \tau \supseteq \sigma' \) we have that \( f(\tau) = 0 \). Note that by definition, every null stable coloring is a 2-coloring. Intuitively, \( f \) is null stable if above each node we can find a node above which \( f \) is constantly 0. Given a finite sequence of null stable colorings, we could sequentially apply the definition of null stable for each coloring and eventually arrive at a node which is colored 0 for all of the colorings. The next theorem shows that the existence of such a node is a disguised form of \( \Sigma^0_2 \) − IND.

**THEOREM 12.** \((\text{RCA}_0)\) The following are equivalent:

1. \( \Sigma^0_2 \) − IND.
2. Suppose \( \langle f_i \rangle_{i<n} \) is a sequence of null stable colorings of \( 2^{<\mathbb{N}} \). Then there is a \( \sigma \in 2^{<\mathbb{N}} \) such that \( f_i(\sigma) = 0 \) for every \( i < n \).

**Proof.** We work in \( \text{RCA}_0 \). First, assume \( \Sigma^0_2 \) − IND and suppose \( \langle f_i \rangle_{i<n} \) is a sequence of null stable 2-colorings. Define \( f : 2^{<\mathbb{N}} \to 2^n \) by \( f(\sigma) = \sum_{i<n} f_i(\sigma) \cdot 2^i \). By Lemma 5 and \( \Sigma^0_2 \) − IND, we may apply \( \text{ECT}(2^{<\mathbb{N}}) \) to \( f \) and find a \( \sigma_0 \) such that \( \forall \alpha \supseteq \sigma_0 \exists \beta \supseteq \alpha \ f(\sigma_0) = f(\beta) \). Suppose, by way of contradiction, that \( f(\sigma_0) \neq 0 \). Fix \( i < n \) such that \( f_i(\sigma_0) = 1 \). Since \( f_i \) is null stable, we can find an \( \alpha_0 \supseteq \sigma_0 \) such that for all \( \beta \supseteq \alpha_0 \), \( f_i(\beta) = 0 \). Since \( \sigma_0 \) was chosen using \( \text{ECT}(2^{<\mathbb{N}}) \), for some \( \beta_0 \supseteq \alpha_0 \), \( f(\sigma_0) = f(\beta_0) \). However, \( f_i(\sigma_0) = 1 \) and \( f_i(\beta_0) = 0 \), so \( f(\sigma_0) \neq f(\beta_0) \), yielding the desired contradiction. Thus we must have \( f(\sigma_0) = 0 \). Since \( f(\sigma_0) = 0 \), \( f_i(\sigma_0) = 0 \) for every \( i < n \).

To prove that item 2 implies item 1, we will use item 2 to deduce TT(1) and apply Theorem 7. Assume item 2 and let \( f : 2^{<\mathbb{N}} \to n \). Define the sequence \( \langle f_i \rangle_{i<n} \) by setting \( f_i(\sigma) = 1 \) if \( f(\sigma) = i \) and \( f_i(\sigma) = 0 \) if \( f(\sigma) \neq i \). If each \( f_i \) was null stable, then by item 2 we could locate a \( \sigma \) such that \( f_i(\sigma) = 0 \) for all \( i < n \), contradicting the fact that \( f(\sigma) = i \) for some \( i < n \). Thus there is an \( i_0 \) such that \( f_{i_0} \) is not null stable. For this \( i_0 \), we can locate a \( \sigma_0 \) such that for every \( \sigma' \supseteq \sigma_0 \) there is a \( \tau \supseteq \sigma' \) such that \( f_{i_0}(\tau) = 1 \). Choose any \( \tau \supseteq \sigma_0 \) with \( f_{i_0}(\tau) = 1 \). Then \( f(\tau) = i_0 \) and every node extending \( \tau \) has an extension of color \( i_0 \). Consequently the standard tree for \( f \) of color \( i_0 \) based at \( \tau \) witnesses TT(1) for \( f \). Since item 2 is \( \Pi^1_1 \), \( \Sigma^0_2 \) − IND follows by Theorem 7. (We could substitute a use of Theorem 11 for Theorem 7 here, if we liked.)
The construction in the preceding proof of a single coloring from many 2-colorings suggests a way to exchange one application of $\text{TT}(1)$ for many colors for many simultaneous applications of $\text{TT}(1)$ restricted to 2-colorings. In fact, we will see that the corresponding formulations are provably equivalent. This is of some interest, since $\text{TT}(1)$ restricted to any standard number of colors is a theorem of $\text{RCA}_0$. The next theorem capitalizes on this notion, and even restricts the simultaneous applications to $\text{TT}(1)$ for null stable 2-colorings, which for single applications is very clearly a theorem of $\text{RCA}_0$.

**THEOREM 13.** ($\text{RCA}_0$) The following are equivalent:

1. $\text{TT}(1)$.

2. Suppose that for each $i < n$, $f_i : 2^{<\omega} \rightarrow 2$. Then there is a subtree $S \subseteq 2^{<\omega}$ order isomorphic to $2^{<\omega}$ such that for each $i < n$, $f_i$ is constant on $S$.

3. Item 2 holds in the case where each $f_i$ is null stable.

**Proof.** Assuming $\text{RCA}_0$, item 2 can be proved by applying $\text{TT}(1)$ to the function $f$ constructed as in the first paragraph of the proof of Theorem 12. Since item 3 is a restricted form of item 2, it remains only to prove $\text{TT}(1)$ from item 3.

Suppose $f : 2^{<\omega} \rightarrow n$. Construct $f_i$ for $i < n$ as in the reversal for Theorem 12. If one of the $f_i$ functions is not null stable, then the standard tree construction as in the proof of the reversal of Theorem 12 witnesses $\text{TT}(1)$ for $f$, completing the proof. If all of the $f_i$ functions are null stable, then we may apply item 3 to find a subtree $S$ isomorphic to $2^{<\omega}$ and monochromatic for all the $f_i$ functions. Let $\sigma$ be the root node of $S$. Now $f(\sigma) = i_0$ for exactly one $i_0 < n$, and for every $\tau$ in $S$, $f_i(\tau) = 1$ if and only if $i = i_0$. Consequently, $f(\tau) = i_0$ for every $\tau \in S$, so $S$ witnesses $\text{TT}(1)$ for $f$. ■

Using the ideas from this section, we close by proving $\text{TT}(1)$ from a stable version of Ramsey’s theorem in trees for pairs and two colors. This is a tree analog of the proof of $\text{RT}(1)$ from $\text{SRT}_2^2$ in [2].

We adopt the notation from [5] for the following. A function on pairs of comparable tree nodes $f : [2^{<\omega}]^2 \rightarrow k$ is said to be $3$-stable if for each $\sigma \in 2^{<\omega}$ there is a $c < k$ such that for every $\sigma' \supseteq \sigma$ there exists $\tau \supseteq \sigma'$ with $f(\sigma, \rho) = c$ for all $\rho \supseteq \tau$. The principle $\text{S}^3\text{TT}_2^2$ asserts that every 3-stable two coloring of comparable pairs from $2^{<\omega}$ has a monochromatic subtree isomorphic to $2^{<\omega}$.

**THEOREM 14.** ($\text{RCA}_0$) $\text{S}^3\text{TT}_2^2$ implies $\text{TT}(1)$. 

Theorem 13. 

(\text{RCA}_0) The following are equivalent:

1. $\text{TT}(1)$.

2. Suppose that for each $i < n$, $f_i : 2^{<\omega} \rightarrow 2$. Then there is a subtree $S \subseteq 2^{<\omega}$ order isomorphic to $2^{<\omega}$ such that for each $i < n$, $f_i$ is constant on $S$.

3. Item 2 holds in the case where each $f_i$ is null stable.

**Proof.** Assuming $\text{RCA}_0$, item 2 can be proved by applying $\text{TT}(1)$ to the function $f$ constructed as in the first paragraph of the proof of Theorem 12. Since item 3 is a restricted form of item 2, it remains only to prove $\text{TT}(1)$ from item 3.

Suppose $f : 2^{<\omega} \rightarrow n$. Construct $f_i$ for $i < n$ as in the reversal for Theorem 12. If one of the $f_i$ functions is not null stable, then the standard tree construction as in the proof of the reversal of Theorem 12 witnesses $\text{TT}(1)$ for $f$, completing the proof. If all of the $f_i$ functions are null stable, then we may apply item 3 to find a subtree $S$ isomorphic to $2^{<\omega}$ and monochromatic for all the $f_i$ functions. Let $\sigma$ be the root node of $S$. Now $f(\sigma) = i_0$ for exactly one $i_0 < n$, and for every $\tau$ in $S$, $f_i(\tau) = 1$ if and only if $i = i_0$. Consequently, $f(\tau) = i_0$ for every $\tau \in S$, so $S$ witnesses $\text{TT}(1)$ for $f$. ■

Using the ideas from this section, we close by proving $\text{TT}(1)$ from a stable version of Ramsey’s theorem in trees for pairs and two colors. This is a tree analog of the proof of $\text{RT}(1)$ from $\text{SRT}_2^2$ in [2].

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**THEOREM 14.** ($\text{RCA}_0$) $\text{S}^3\text{TT}_2^2$ implies $\text{TT}(1)$.
Proof. Assume $\text{RCA}_0$ and suppose $f : 2^{<\mathbb{N}} \to \mathbb{N}$. Define $g : [2^{<\mathbb{N}}]^2 \to 2$ for $\sigma \subset \tau$ by setting $g(\sigma, \tau) = 1$ if and only if $f(\sigma) = f(\tau)$. If $\sigma$ witnesses that $g$ is not 3-stable, then the standard tree for $f$ based at $\sigma$ with color $f(\sigma)$ witnesses $TT(1)$ for $f$. If $g$ is 3-stable, we may apply $S^3TT_2$ to find a monochromatic subtree $S$ for $g$. Select a sequence of $n + 1$ comparable nodes in $T$. Some pair in this sequence must be colored identically by $f$. Thus $g([S]^2) \equiv 1$, and $S$ is a monochromatic subtree for $f$. ■

It is not known whether or not $\text{SRT}_2^2$ implies $\Sigma^0_3 - \text{IND}[2]$. Similarly, it is not known whether or not $S^3TT_2$ implies $\Sigma^0_3 - \text{IND}$. Since $S^3TT_2$ is a $\Pi^1_3$ sentence, Theorem 7 is not applicable. Thus, Theorem 14 is a candidate for a proof of $TT(1)$ that does not rely on a disguised use of $\Sigma^0_3 - \text{IND}$.

Bibliography


