Reverse mathematics and colorings of hypergraphs

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Abstract

Working in subsystems of second order arithmetic, we formulate several representations for hypergraphs. We then prove the equivalence of various vertex coloring theorems to WKL_0, ACA_0, and Π^1_1-CA_0. MSC: 03B30; 03F35

A hypergraph consists of a set of vertices together with a set of edges. While edges in graphs always have exactly two vertices, hypergraphs may have edges of any cardinality, finite or infinite. Some authors exclude edges that are empty of have a single vertex, but for this article, edges may be of any size. After considering alternatives for the representations of edges, we will turn to results on the strength of theorems related to three types of vertex colorings.

Representations of edges

The edges of hypergraphs can be presented in a variety of ways. If all the edges are finite, each edge can be encoded by a single number. In this case, the edges can be represented by a set of codes or by a sequence of codes. Finite edges and infinite edges can both be represented by a sequence of characteristic functions. Some changes in representation can be carried out in RCA_0, while others require additional set comprehension, as shown by the next four results.

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Theorem 1. (RCA_0) *If H is a hypergraph with finite edges represented by a set of edges, then there is a hypergraph H' with exactly the same edges represented by a sequence of its edges, possibly with repetitions.*

Proof. We argue in RCA_0. Suppose $H = \langle V, E\rangle$ is a hypergraph with finite edges, where E is a set of integer codes for the edges of H. Let e_0 be an integer code for an edge of H. Define the function $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \begin{cases} n & n \in E \\ e_0 & \text{otherwise} \end{cases}$$

Then the range of f is E, and V together with f represents H using a sequence of codes. □

Theorem 2. (RCA_0) *If H is a hypergraph with finite edges represented by a sequence of its edges, then there is a hypergraph H' with exactly the same edges represented by a sequence of characteristic functions for its edges.*

Proof. We argue in RCA_0. Suppose $H = \langle V, E\rangle$ is a hypergraph with finite edges, where E is the sequences for the edges of H. Define the characteristic function χ_i by

$$\chi_i(v_j) = \begin{cases} 1 & v_j \in \langle e_i\rangle \\ 0 & \text{otherwise} \end{cases}$$

By recursive comprehension, the sequence of characteristic function $\langle \chi_i\rangle_{i\in\mathbb{N}}$ exists. The vertices V together with the sequence of characteristic functions for each edge represent H. □

Theorem 3. (RCA_0) *The following are equivalent:*

(1) ACA_0.

(2) *If H is a hypergraph with finite edges represented by a sequence of edges, then H can be represented by a set of edges.*

Proof. First we will prove that (1) implies (2). Reasoning in ACA_0, let H be a hypergraph with finite edges represented by the sequence $\langle e_i\rangle_{i\in\mathbb{N}}$. Arithmetical comprehension can prove the edge set $\{e \mid \exists i(e = e_i)\}$ exists.

To prove the converse, by Lemma III.1.3 of Simpson [7], it suffices to use item (2) to prove the existence of the range of an injection. Let $g : \mathbb{N} \to \mathbb{N}$ be an injection. Define the hypergraph $H = \langle V, E\rangle$ as follows. Let $\mathbb{N}$ be the set of vertices. Define a sequence of edges $\langle e_n\rangle_{n\in\mathbb{N}}$ by $e_n = \{0, g(n)+1\}$. Note that m is in the range of g if and only if the set $\{0, m+1\}$ is an edge of H. Given the set of edges of H, recursive comprehension proves the existence of the range of g. $\square$

Theorem 4. (RCA_0) *The following are equivalent:*

(1) ACA_0.

(2) *If H is a hypergraph with finite edges represented by a sequence of characteristic functions, then H can be represented by a sequence of finite set codes for edges.*

Proof. We begin by proving that 1 implies 2. Reasoning in ACA_0, let H be a hypergraph with finite edges represented by the sequence of characteristic functions $\langle e_i\rangle_{i\in\mathbb{N}}$. Define $s_i = \{j \mid e_i(j) = 1\}$ for $i \in \mathbb{N}$. Then the sequence $\langle s_i\rangle_{i\in\mathbb{N}}$ is arithmetically definable.

To prove the converse, it suffices to use statement 2 to prove the existence of a range of an injection. Let $g : \mathbb{N} \to \mathbb{N}$ be that injection. Let H be the hypergraph with vertex set $\mathbb{N}$ and edges defined by the sequence of characteristic functions $\langle e_i\rangle_{i\in\mathbb{N}}$ defined as follows

$$e_i(n) = \begin{cases} 1 & n = 2i \vee n = 2i+2 \\ 0 & n = 2j \wedge j \notin \{i, i+1\} \\ 1 & n = 2j+1 \wedge g(j) = i \\ 0 & n = 2j+1 \wedge g(j) \neq i \end{cases}$$

The recursive comprehension axiom proves the existence of $\langle e_i\rangle$ using g as a parameter. Applying the principle of statement 2, let $\langle s_i\rangle_{i\in\mathbb{N}}$ be a sequence of finite set codes for the edges of H. Given any value y, successively examine the set codes until we locate the unique s_i encoding a set containing $2y$ and $2y+2$. If this set contains only $2y$ and $2y+2$, then y is not in the range of g, but if the set also contains another element, then y is in the range of g. Thus, recursive comprehension proves the existence of the range of g. $\square$

In order to state compactness results for vertex colorings, we need to consider finite substructures of hypergraphs. The following terminology is based on that of Berge [1]. When constructing substructures of a hypergraph H, one can either require that all edges in the substructure are edges of H, or allow edges of the substructure to be subsets of edges of H.

Definition. A *partial hypergraph* of a hypergraph H consists of a subset V of the vertices of H and a subset E of the edges of H such that for every edge $e \in E$, $e \subset V$.

Definition. A *partial subhypergraph* of a hypergraph H consists of a subset V of the vertices of H and a set E of subsets of V such that for every $e \in E$, there is an edge $e' \in H$ such that $e = e' \cap V$.

Vertex colorings and finite edges

A vertex coloring is a map from the vertices of a hypergraph into a set of colors, usually coded by a subset of $\mathbb{N}$. In the literature, most definitions for vertex colorings are stated for finite hypergraphs, but those listed below extend naturally to infinite hypergraphs. The definition for conflict-free colorings is based on that of Smorodinsky [8]. The other definitions can be found in the book of Berge [1].

Definition. Suppose $\langle V, E\rangle$ is a hypergraph. Let $\chi : V \to \mathbb{N}$ be a coloring of the vertices. We say:

(1) χ is a k-coloring if the range of χ is contained in $\{0, 1, \ldots k-1\}$.

(2) χ is *proper* if χ is non-constant on each edge that contains more than one vertex.

(3) χ is *strong* if χ is injective on each edge.

(4) χ is *conflict-free* [8] if each edge contains one vertex whose color matches no other vertex in the edge. That is, $\forall e \exists j \ |\{v \in e \mid \chi(v) = j\}| = 1$.

Theorem 3.4 of Hirst [4] states that WKL_0 is equivalent to the statement that every locally k-colorable graph has a k-coloring. The following result generalizations this theorem to the hypergraph setting, for hypergraphs with sequences or sets of finite edges.

Theorem 5. (RCA_0) *For $k \geq 2$, the following are equivalent:*

(1) WKL_0.

(2) *Let H be a hypergraph with a sequence of finite edges. If every finite partial hypergraph of H has a proper k-coloring, then H has a proper k-coloring.*

(3) *Statement* (2) *with "proper" replaced by "conflict-free".*

(4) *Statement* (2) *with "sequence of edges" replaced by "set of edges."*

(5) *Statement* (3) *with "sequence of edges" replaced by "set of edges."*

Proof. To prove that (1) implies (2), assume WKL_0 and let H be a hypergraph with vertex set $\{v_1, v_2, \ldots\}$ such that every finite partial hypergraph of H has a proper k-coloring. For every n, let m_n be the least integer such that $m_n > m_{n-1}$ and if a vertex v_i appears in one of the first n edges of H then $i \leq m_n$. Let H_n be the finite partial hypergraph with vertex set $\{v_1, v_2, \ldots, v_{m_n}\}$ and the first n edges of H. Let T be the tree consisting of those finite sequences σ in $k^{<\mathbb{N}}$ such that whenever $\text{length}(\sigma) = m_n$, σ is a proper k-coloring of H_n. Since every finite partial hypergraph of H has a proper k-coloring, T must contain infinitely many sequences. By WKL_0, T has an infinite path. This path yields a proper k-coloring of H.

To prove that (1) implies (3), repeat the previous proof replacing every instance of the word "proper" by "conflict-free." Theorem 1 shows that (2) implies (4) and (3) implies (5) It remains to show that both (4) and (5) imply (1). For graphs, that is, for hypergraphs in which every edge has exactly two vertices, every coloring is proper if and only if it is conflict-free. Thus, both (4) and (5) imply that if every finite subgraph of a graph H is k-colorable, then H can be k-colored. This implies WKL_0 by Theorem 3.4 of Hirst [4]. $\square$

For the analog of the previous theorem for graphs with edges represented by sequences of characteristic functions, additional set comprehension strength is required.

Theorem 6. (RCA_0) *For any $k \geq 2$, the following are equivalent.*

(1) ACA_0.

(2) *Suppose H is a hypergraph with finite edges given by a sequence of characteristic functions. If every finite partial hypergraph of H has a proper k-coloring then H has a proper k-coloring.*

(3) *Statement* (2) *with "proper" replaced by "conflict-free."*

Proof. To prove that (1) implies (2), fix k, assume ACA_0, and suppose H is a hypergraph with finite edges given by a sequence of characteristic functions.

Further suppose that every finite partial hypergraph of H can be properly k-colored. By Theorem 4, we can find the sequence of edges for H. By Theorem (5) and the fact that ACA_0 implies WKL_0, H has a proper k-coloring.

The proof that (1) implies (3) is similar to the preceding argument, with proper replaced by conflict-free.

For the reversals, we begin by considering the proof that (2) implies (1) for $k = 2$. By Lemma III.1.3 of Simpson [7], it suffices to use RCA_0 and (2) to prove the existence of the range of an arbitrary injection f. The vertices of our hypergraph H will be $\{b_i \mid i \in \mathbb{N}\} \cup \{v_{i,j} \mid i, j \in \mathbb{N}\}$. The edges of H will be represented by characteristic functions for the following sets. For all n and i, $p_{n,i} = \{v_{n,i}, v_{n,i+1}\}$, $q_n = \{b_n, b_{n+1}\}$, $r_n = \{b_{2n}, v_{f(n),2n}\}$, and $s_n = \{v_{n,0}, b_{2n+1}\} \cup \{b_{2i} \mid f(i) = n\}$. Because f is an injection, b_{2i} is in s_n if and only if $\exists t < 2i(f(t) = n)$, so the characteristic function for s_n is uniformly computable from f. Via dovetailing, recursive comprehension suffices to prove the existence of a sequence of characteristic functions for all these edges of H. Note that edges of the form s_n contain two vertices if n is not in the range of f and three vertices if n is in the range. All other edges contain exactly two vertices.

We claim that every finite partial subhypergraph of H has a proper 2-coloring. Let G be a partial subhypergraph. Let n_0 be the maximum natural number such that b_{2n_0} is in G. Define the coloring χ by $\chi(b_n) = n \mod 2$ if $n \leq 2n_0 + 1$, $\chi(v_{n,j}) = j \mod 2$ if $\forall t \leq n_0(f(t) \neq n)$, and $\chi(v_{n,j}) = j + 1 \mod 2$ if $\exists t \leq n_0\ (f(t) = n)$. The first clause guarantees that χ is a proper coloring with regard to every edge of the form q_n, and the last two clauses insure that χ is proper on edges of the forms $p_{n,i}$, r_n and s_n restricted to the vertices of G.

Apply (2) to obtain a proper 2-coloring of H. If necessary, permute the colors so that $\chi(b_0) = 0$. Fix n. If $f(t) = n$ for some t, then $r_t = \{b_{2t}, v_{n,2t}\}$ is in H, so $\chi(v_{n,0}) = 1$. If n is not in the range of f, then $s_n = \{v_{n,0}, b_{2n+1}\}$ is in H, so $\chi(v_{n,0}) = 0$. Thus the range of f is defined by $\{n \mid \chi(v_{n,0}) = 1\}$, which exists by recursive comprehension.

The proof that (2) implies (1) can be extended to values of k greater than 2 by adding a complete graph on $k - 2$ vertices to H and connecting each vertex in the complete graph to every vertex in the prior construction. Finally, for graphs with edges of size at most 3, any coloring is conflict-free if and only if it is proper. Thus (3) implies (1) can be proved by replacing all uses of "proper" in the preceding arguments with "conflict-free." □

For strong colorings, edge representation does not affect the strength of the coloring statements. Additionally, in some settings, finite colorability suffices to imply WKL_0 over RCA_0. This provides an interesting contrast to the case for graphs, as described following the proof of the next result.

Theorem 7. (RCA_0) *The following are equivalent:*

(1) WKL_0.

(2) *Let H be a hypergraph with any edge representation. If there is a k such that every finite partial hypergraph of H has a strong k-coloring, then H has a strong k-coloring.*

(3) *Let H be a hypergraph with a set of finite sets for edges. If every finite partial hypergraph of H has a strong 3-coloring, then H has a strong k-coloring for some k.*

(4) *Let H be a hypergraph with a sequence of finite sets for edges. If every finite partial hypergraph of H has a strong 2-coloring, then H has a strong k-coloring for some k.*

Proof. To prove that (1) implies (2), assume WKL_0 and let H be a hypergraph. By Theorem 1 and Theorem 2, we may assume that the edges of H are given by a sequence of characteristic functions. Fix k and suppose that for every n, the partial subhypergraph given by the first n values of each of the the first n characteristic functions has a strong k-coloring. Let T be the tree consisting of those finite sequences σ in $k^{<\mathbb{N}}$ such that σ is a strong k-coloring of the partial subhypergraph defined by the first length(σ) many values of the first length(σ) many edge characteristic functions. Because the finite partial hypergraphs of H are colorable, T must contain infinitely many sequences. By WKL_0, T has an infinite path. This path yields a strong k-coloring of H.

Note that (3) follows immediately from from (2) restricted to $k = 3$. To prove that (3) implies (1), by Lemma IV.4.4 of Simpson [7], it suffices to use (3) to separate the ranges of injections with disjoint ranges. Fix injections f and g such that $\forall m \forall n (f(m) \neq g(n))$ and construct a hypergraph H as follows. The vertices of H consist of the sets $U = \{u_i \mid i \in \mathbb{N}\}$ and $V = \{v_i \mid i \in \mathbb{N}\}$. For each triple i, j, and k, the edge $\{u_i, u_j, v_k\}$ is in H if and only if $i < k$, $j < k$, $\exists t < k\ f(t) = i$, and $\exists t < k\ g(t) = j$. Note that each edge contains exactly three vertices and exactly one vertex from V. No edge contains two vertices from U indexed by elements of the range of f, and similarly, pairs of vertices from U indexed by elements of the range of g are forbidden within an edge. Let H_0 be a finite partial hypergraph consisting of vertices $\{u_{i_0}, \dots u_{i_m}\}$ and $\{v_{j_0}, \dots v_{j_n}\}$, and some or all of the edges of H with vertices entirely contained in these sets. The coloring that assigns color 0 to all vertices u_{i_k} such that $\exists t < j_n\ f(t) = i_k$, color 1 to all vertices u_{i_k} such that $\exists t < j_n\ g(t) = i_k$, and color 3 to all remaining vertices is a strong 3-coloring of H_0. Thus, every finite partial hypergraph of H has a strong 3-coloring. Apply (3) to obtain a k-coloring of H. Note that if $\exists t\ f(t) = i$ and $\exists t\ g(t) = j$, then u_i and u_j are included in an edge, and so u_i and u_j must have distinct colors. By bounded Σ^0_1 comprehension (which is provable in RCA_0 by Remark II.3.11 of Simpson [7]) the set $C = \{i < k \mid \exists t\ \chi(u_{f(t)}) = i\}$ exists. RCA_0 proves that the set $\{j \in \mathbb{N} \mid \chi(u_j) \in C\}$ exists, contains the range of f, and is disjoint from the range of g.

Item (4) also follows immediately from (2). As in the previous paragraph, to prove that (4) implies WKL_0, let f and g be injections with disjoint ranges. Construct H as follows. The vertices of H are $V = \{v_i \mid i \in \mathbb{N}\}$. Using a bijective pair encoding, for each $i \in \mathbb{N}$, let i_0 and i_1 denote the components of the pair encoded by i. For each i, define the edge $e_i = \{v_{f(i_0)}, v_{g(i_1)}\}$. Let E_0 be any finite set of edges with indices bounded by b, and let V_0 be some set of vertices containing all the vertices in edges in E_0. The coloring which assigns color 0 to all vertices of the form $v_{f(t)}$ for $t < b$ and color 1 to all other vertices of V_0 is a strong 2-coloring of the finite partial hypergraph defined by E_0 and V_0. By (4), H has a strong k-coloring for some k; call it χ. By bounded Σ^0_1 comprehension, provable in RCA_0 by Exercise II.3.13 of Simpson [7], the set $K = \{j < k \mid \exists t\ \chi(v_{f(t)}) = j\}$ exists. By recursive comprehension, the set $S = \{n \mid \chi(v_n) \in K\}$ exists. By the construction of H, S contains all elements of the range of f and excludes all elements of the range of g. □

In the preceding theorem, if the number 3 is replaced by 2 in item (3), the resulting statement asserts that every locally 2-colorable graph has a finite coloring. This statement is known to be equivalent to WKL_0 over RCA_0 plus Σ^0_2 induction, but the equivalence over RCA_0 is one of many open questions listed in section §5 of Dorais, Hirst, and Shafer [3]. Item (4) shows that for graphs with edges presented as a sequence rather than as a set, the statement that every locally 2-colorable graph has a finite coloring is equivalent to WKL_0 over RCA_0.

Vertex colorings and infinite edges

Allowing hypergraphs with infinite edges significantly affects the nature of vertex colorings. Clearly, any strong vertex coloring of a hypergraph with an infinite edge must use infinitely many colors. Hypergraphs with infinite edges may or may not have finite proper or conflict-free colorings. We will show that sorting those graphs with finite colorings from those without requires Π^1_1-CA_0. The next definition and theorem assist in that proof.

Definition. If T is a tree, the set L of *leaves of* T consists of those sequences in T which have no extensions. That is, $L = \{\sigma \in T \mid \forall n\ \sigma^\frown n \notin T\}$.

Lemma 8. (RCA_0) *The follow are equivalent.*

(1) Π^1_1-CA_0.

(2) *If $\langle T_i \rangle_{i\in\mathbb{N}}$ is a sequence of trees in $\mathbb{N}^{<\mathbb{N}}$, then there is a function $f : \mathbb{N} \to 2$ such that $f(i) = 1$ if and only if T_i contains an infinite path.*

(3) *If $\langle T_i \rangle_{i\in\mathbb{N}}$ is a sequence of trees and $\langle L_i \rangle_{i\in\mathbb{N}}$ is a sequence of sets such that for each i, L_i is the set of leaves of T_i, then there is a function $f : \mathbb{N} \to 2$ such that $f(i) = 1$ if and only if T_i contains an infinite path.*

Proof. We work in RCA_0. The equivalence of (1) and (2) is Lemma VI.1.1 of Simpson [7]. Because (3) is a special case of (2), we need only prove that (3) implies (2).

Let $\langle T_i \rangle_{i\in\mathbb{N}}$ be a sequence of trees as in (2). For any sequence σ of positive integers, let σ_{-1} denote the sequence of the same length as σ such that for all i, $\sigma_{-1}(i) = \sigma(i) - 1$. For each i, define the tree $\hat{T}_i$ by letting $\tau \in \hat{T}_i$ if and only if either $\tau_{-1} \in T_i$ or $\tau = \sigma^\frown 0$ and $\sigma_{-1} \in T_i$. For each i, the leaf set of $\hat{T}_i$ consists precisely of those sequences of the form $\sigma_{-1}{}^\frown 0$ such that $\sigma \in T_i$. RCA_0 can prove that $\langle \hat{T}_i \rangle_{i\in\mathbb{N}}$ and $\langle L_i \rangle_{i\in\mathbb{N}}$ exist. Note that p is an infinite path in $\hat{T}_i$ if and only if p_{-1} is an infinite path in T. Thus, the function f satisfying (3) for $\langle \hat{T}_i \rangle_{i\in\mathbb{N}}$ also satisfies (2) for $\langle T_i \rangle_{i\in\mathbb{N}}$. □

The preceding result allows us to prove the reversal below in a single step, rather than using (2) to prove ACA_0 and then deducing Π^1_1-CA_0 in a second step.

Theorem 9. (RCA_0) *For each $k \geq 2$, the following are equivalent.*

(1) Π^1_1-CA_0.

(2) *If* $\langle H_i \rangle_{i \in \mathbb{N}}$ *is a sequence of hypergraphs, then there is a function* $f : \mathbb{N} \to 2$ *such that* $f(i) = 1$ *if and only if* H_i *has a proper* k*-coloring.*

(3) *Statement* (2) *with "proper" replaced by "conflict-free."*

Proof. Assume RCA_0. To prove that (1) implies (2), fix $k \geq 2$ and suppose $\langle H_i \rangle_{i \in \mathbb{N}}$ is a sequence of hypergraphs. Assuming that the vertices of H are a subset of $\mathbb{N}$, the statement "H has a proper k-coloring" asserts the existence of a function $g : \mathbb{N} \to k$ such that g is not constant on any edge. Thus, the set of indices i such that H_i has a proper k-coloring is definable by a Σ^1_1 formula. Applying Π^1_1-CA_0, the complement of this set of indices exists, so by recursive comprehension, the set of indices and its characteristic function also exist. This function satisfies item (2) of the theorem. A similar argument shows that (1) implies (3).

Next, we will prove that (2) implies (1) for the case $k = 2$, indicating parenthetically how to modify the argument for (3) implies (1). By Lemma 8, it suffices to use (2) to determine which trees are well-founded in a list of trees with leaf sets. Let $\langle T_i \rangle_{i \in \mathbb{N}}$ be a sequence of trees in $\mathbb{N}^{<\mathbb{N}}$, and let $\langle L_i \rangle_{i \in \mathbb{N}}$ be the corresponding leaf sets. Given any tree $T \subset \mathbb{N}^{\mathbb{N}}$ with leaf set L, define a hypergraph H as follows. The vertices of H are $\{a_0, a_1, b_0, b_1, s\}$ together with vertices labeled σ_0 and σ_1 for each nonempty sequence σ in T. (One can routinely assign integer codes to these vertices, and arrange for the set of codes to be $\mathbb{N}$ or an initial segment of $\mathbb{N}$, if so desired.) The edges of H consist of

- (a_0, a_1), (a_1, s), (b_0, b_1), and (b_1, s),
- (σ_0, σ_1) for every nonempty $\sigma \in T$,
- (σ_1, s) if σ is a leaf of T,
- $E_\sigma = \{\sigma_1\} \cup \{\tau_0 \mid \tau \in T \wedge \exists n\ \tau = \sigma^\frown n\}$ if $\sigma \in T$ is not a leaf, and
- $E_0 = \{a_0, b_0\} \cup \{\sigma_0 \mid \sigma \in T \wedge \text{length}(\sigma) = 1\}$.

Using T and L as parameters, RCA_0 can prove the existence of H uniformly. Note that E_0 and E_σ may be infinite edges.

Suppose $c : \mathbb{N} \to \{\text{red}, \text{ blue}\}$ is any 2-coloring of the vertices of H. Swapping colors if necessary, let s be blue. Assume c is proper. Then a_1 and b_1 are red and a_0 and b_0 are blue. Also, for any $\sigma \in T$ if σ_0 is red, then σ_1

is blue, so $\{\sigma_1, s\}$ is not an edge and σ is not a leaf. Because σ_1 is blue, by the definition of E_σ, for some immediate successor τ of σ, τ_0 must be red. Finally, to properly color E_0, for some $\sigma \in T$ of length 1, σ_0 is red. Let n be the least value such that for $\tau = \sigma^\frown n$, τ_0 is red. Iterating this process traces an infinite path in T. Summarizing, if c properly 2-colors H, then T has an infinite path. (Every conflict-free coloring is proper, so if c is conflict-free then T has an infinite path.)

Conversely, suppose that T has an infinite path p, and let $\sigma^0 \subset \sigma^1 \subset \sigma^2 \subset \ldots$ be the nonempty initial segments of p. Define c by $c(s) = c(a_0) = c(b_0) = \text{blue}$, $c(a_1) = c(b_1) = \text{red}$, $\sigma^i_0 = \text{red}$ and $\sigma^1_1 = \text{blue}$ for σ^i in p, and $\tau_0 = \text{blue}$ and $\tau_1 = \text{red}$ for $\tau \in T$ not in p. Treating the definitions of the edges of H as cases, one can verify that c is a proper (conflict-free) 2 coloring of H. Summarizing the last two paragraphs, H has a proper (conflict-free) 2-coloring if and only if T has an infinite path.

Carrying out the the construction uniformly for all the trees in $\langle T_i \rangle_{i \in \mathbb{N}}$ and applying (2) of the theorem, we can find the characteristic function for the well-founded trees, as desired. The parenthetical comments show that for $k = 2$, (3) implies (1).

For values of $k > 2$, modify the construction of the previous reversal by adding a complete graph with $k - 2$ vertices to H, connecting each vertex of the complete graph to every vertex of H by an edge consisting of two vertices. The resulting hypergraph has a proper (conflict-free) k-coloring if and only if T is not well-founded. $\square$

It would be interesting to know if the preceding result continues to hold if "k-coloring" is replaced by "finite coloring." Lemma 8 may prove useful in recasting the preceding theorem as a result related to Σ^1_1 completness using many-one reducibility, or as a result on Weihrauch reducibility.

Infinite edges can interfere with finite conflict-free colorings of hypergraphs. The following graph illustrates this situation.

Definition. The $\mathcal{M}$-graph (the Matryoshka graph) is the hypergraph with vertex set $\mathbb{N}$ and edges $\{E_j \mid j \in \mathbb{N}\}$ where $E_j = \{k \mid j \leq k\}$.

Every finite partial subhypergraph of the $\mathcal{M}$-graph has a conflict-free 2-coloring. For example, given any finite collection of vertices, simply color the largest numbered vertex red and all other vertices blue. On the other hand, the entire $\mathcal{M}$-graph has no finite conflict-free coloring. This can be proved directly by induction on Σ^0_2 formulas, or by using the following lemma.

Lemma 10. (RCA_0) *The following are equivalent.*

(1) ERT *(eventually repeating tails): Suppose $f : \mathbb{N} \to k$ for some $k \in \mathbb{N}$. Then there is a $b \in \mathbb{N}$ such that for all $x \geq b$ there is a $y \geq b$ such that $x \neq y$ and $f(x) = f(y)$.*

(2) *No finite coloring of the $\mathcal{M}$-graph is conflict-free.*

Proof. We will work in RCA_0. To prove that (1) implies (2), suppose that $f : \mathbb{N} \to k$ is a finite coloring of the $\mathcal{M}$-graph. Apply ERT to find b such that for all $x \geq b$ there is a $y \geq b$ such that $y \neq x$ and $f(x) = f(y)$. Then every color appearing in E_b appears at least twice. Thus f is not conflict-free.

To prove the converse, let $f : \mathbb{N} \to k$ be any function. We can view f as a k-coloring of the $\mathcal{M}$-graph. By (2), f is not conflict-free. Thus there is an edge E_b such that every color appearing in b appears at least twice. Thus b witnesses that ERT holds. □

The principle ERT follows trivially from the principle ECT which asserts that if $f : \mathbb{N} \to k$ then there is a b such that for all $x \geq b$ there are infinitely many values of y such that $f(x) = f(y)$. By Theorem 6 of Hirst [5], ECT is equivalent to Σ^0_2 induction. Thus, $\mathsf{I}\Sigma^0_2$ suffices to prove that no finite coloring of the $\mathcal{M}$-graph is conflict-free.

We will show that this use of induction is not necessary by proving ERT from SRT^2_2, the stable Ramsey theorem for pairs and two colors. By Corollary 2.6 of Chong, Slaman, and Yang [2], SRT^2_2 cannot prove $\mathsf{I}\Sigma^0_2$, so neither can ERT.

Theorem 11. (RCA_0) SRT^2_2 *implies* ERT*.*

Proof. Let $f : \mathbb{N} \to k$. Define $g : [\mathbb{N}]^2 \to 2$ by $g(a, b) = 1$ if and only if for some $x \in [a, b)$, $f(x)$ appears exactly once in the range of f restricted to $[a, b)$. Because the range of f is k, for fixed a and increasing x, the value of $g(a, x)$ can only change at most $2k$ times. Thus g is a stable coloring.

Apply SRT^2_2 to g to obtain an infinite set $H = \{x_0, x_1, x_2 \dots\}$ that is monochromatic for g. Suppose, by way of contradiction, that $g([H]^2) \equiv 1$. Let H_0 consist of the first $3 \cdot 2^{k-1}$ elements of H. The elements of H_0 define $3 \cdot 2^{k-1} - 1$ consecutive half-open intervals. Because $g(x_0, x_{3 \cdot 2^{k-1}}) = 1$, some value of f appears exactly once in $[x_0, x_{3 \cdot 2^{k-1}})$. Call this k_0. Either on the left or the right of this location, there must be a collection of $3 \cdot 2^{k-2}$ consecutive elements of H_0. Call these H_1. Note that the range of f on the intervals defined by H_1 must omit k_0. Thus, the range of f on these intervals contains at most $k-1$ elements. Iterating this construction, H_{k-1} will consist of three consecutive elements of H_0 such that f is constant on the union of the two associated subintervals. No value of f appears exactly once in the range of f on this union, contradicting the assumption that g applied to the endpoints yields 1. Thus $g([H]^2) \equiv 0$ and every value of f appearing at or after x_0 must appear at least twice. $\square$

It would be interesting to know how ERT compares in strength to other statements that are weaker than $\mathsf{I}\Sigma^0_2$, for example, like those described by Kreuzer and Yokoyama [6]

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