Reverse Mathematics: Constructivism and Combinatorics

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OCSA outcomes: Publications


OCSA outcomes: Presentations


- *Reverse mathematics and persistent reals*, given in the Midwest Computability Seminar X at the University of Chicago on November 1, 2011.

- *Two familiar principles in disguise*, given in the Notre Dame Logic Seminar on October 27, 2011.

- *Two combinatorial proofs and some related questions*, given at the Reverse Mathematics Workshop at the University of Chicago on September 17, 2011.

- *Reverse Mathematics: Constructivism and Combinatorics*, given at Foundational Questions in the Mathematical Sciences, a meeting sponsored by the John Templeton Foundation at the International Academy Traunkirchen, Austria on July 8-12, 2011.
A weak form of Hindman’s theorem

HIL: Suppose \( f : \mathbb{N}^{< \mathbb{N}} \to k \) is a finite coloring of the finite subsets of the natural numbers. Then there is an infinite sequence \( \langle X_i \rangle_{i \in \mathbb{N}} \) of distinct finite sets and a color \( c < k \) such that for every finite set \( F \subset \mathbb{N} \) we have \( f(\bigcup_{i \in F} X_i) = c \).

HTU: Suppose \( f : \mathbb{N}^{< \mathbb{N}} \to k \) is a finite coloring of the finite subsets of the natural numbers. Then there is an infinite sequence \( \langle X_i \rangle_{i \in \mathbb{N}} \) of increasing finite sets and a color \( c < k \) such that for every finite set \( F \subset \mathbb{N} \) we have \( f(\bigcup_{i \in F} X_i) = c \).

\( X_i < X_j \) means \( \max(X_i) < \min(X_j) \)

Note: This material arose from discussions with Henry Towsner.
An example of reverse mathematics

Theorem

(RCA₀) The following are equivalent:

1. HIL :: If \( f : \mathbb{N}^{<\mathbb{N}} \to k \) then there is a color \( c \) and a sequence \( \langle X_i \rangle_{i \in \mathbb{N}} \) of distinct sets such that all finite unions of sets in the sequence have the color \( c \).

2. RT(1) :: If \( f : \mathbb{N} \to k \) then there is a \( c < k \) such that \( \{ n \mid f(n) = c \} \) is infinite.

RCA₀ is a set of axioms that talk about natural numbers and sets of naturals numbers. The axioms include ordered semi-ring axioms for \( \mathbb{N} \), induction for formulas with only one (number) quantifier, and a set-existence axiom for computable subsets of \( \mathbb{N} \). Many theorems can be expressed in RCA₀, but only some can be proved in the system.
Theorem (RCA₀) The following are equivalent:

1. HIL: If \( f : \mathbb{N}^{<\mathbb{N}} \to k \) then there is a color \( c \) and a sequence \( \langle X_i \rangle_{i \in \mathbb{N}} \) of distinct sets such that all finite unions of sets in the sequence have the color \( c \).

2. RT(1): If \( f : \mathbb{N} \to k \) then there is a \( c < k \) such that \( \{ n \mid f(n) = c \} \) is infinite.

Sketch.

(1) \( \Rightarrow \) (2). Given \( f : \mathbb{N} \to k \), define \( g(X) = f(\max(X)) \). Apply HIL. \( f \) is constant on the infinite set of maxima.
Theorem (RCA₀) The following are equivalent:

1. HIL : If $f : \mathbb{N}^{\triangleleft} \rightarrow k$ then there is a color $c$ and a sequence $\langle X_i \rangle_{i \in \mathbb{N}}$ of distinct sets such that all finite unions of sets in the sequence have the color $c$.

2. RT(1) : If $f : \mathbb{N} \rightarrow k$ then there is a $c < k$ such that $\{n \mid f(n) = c\}$ is infinite.

Sketch.

(1) $\rightarrow$ (2). Given $f : \mathbb{N} \rightarrow k$, define $g(X) = f(\text{max}(X))$. Apply HIL. $f$ is constant on the infinite set of maxima.

(2) $\rightarrow$ (1). Given $f : \mathbb{N}^{\triangleleft} \rightarrow k$, define $g(n) = f([0, n])$. Apply RT(1) to find $n_0, n_1, \ldots$ monochromatic. Let $X_i = [0, n_i]$.  □
Why bother?

Based on Tait’s analysis of Hilbert’s program, Simpson [5] says that a theorem is *finitistically reducible* if it is provable in a theory which is a conservative extension of PRA (primitive recursive arithmetic) for $\Pi^0_1$ sentences.

\[ WKL_0 + RT(1) \text{ is conservative over PRA for } \Pi^0_2 \text{ formulas.} \]

Since $WKL_0 + RT(1)$ proves $RCA_0 + HIL$, we know $HIL$ is finitistically reducible.
Based on Tait’s analysis of Hilbert’s program, Simpson [5] says that a theorem is *finitistically reducible* if it is provable in a theory which is a conservative extension of PRA (primitive recursive arithmetic) for $\Pi^0_1$ sentences.

$WKL_0 + RT(1)$ is conservative over PRA for $\Pi^0_2$ formulas. Since $WKL_0 + RT(1)$ proves $RCA_0 + HIL$, we know $HIL$ is finitistically reducible.

$RCA_0 + HTU$ proves $ACA_0$ [1], so $RCA_0 + HTU$ proves $\Pi^0_1$ formulas that PRA can’t.

The consistency of PRA is a $\Pi^0_1$ formula. $HTU$ is not finitistically reducible.
Dichotomy is provable in RCA$_0$,

**Theorem**
(RCA$_0$) *If $\alpha$ is a real number, then $\alpha \geq 0$ or $\alpha \leq 0$.*

but sequential dichotomy is not...

**Theorem**
(RCA$_0$) *The following are equivalent:*

1. WKL$_0$ (*Infinite 0-1 trees have infinite paths.*)
2. If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of reals, then there is a set $I$ such that for all $i$, $i \in I$ implies $\alpha_i \geq 0$ and $i \notin I$ implies $\alpha_i \leq 0$.

Note: This material is joint work with François Dorais and Paul Shafer [2].
Since RCA<sub>0</sub> proves that sequential dichotomy implies WKL<sub>0</sub>, RCA<sub>0</sub> cannot prove sequential dichotomy.

By a result of Hirst and Mummert [4], since RCA<sub>0</sub> cannot prove sequential dichotomy, E-HA<sub>ω</sub> + AC + IP<sub>ω</sub> ef does not prove dichotomy.

(E-HA<sub>ω</sub> is an axiomatization of constructive analysis, AC is a choice scheme, and IP<sub>ω</sub> ef is an independence of premise scheme for ∃-free formulas.)

The result from [4] is not a biconditional, but a computable restriction of sequential dichotomy can indicate a candidate for a constructive restriction of dichotomy.
Definition: A real $\alpha$ is persistent if

- $\forall s (\alpha(s) \geq 0 \rightarrow \exists t (t > s \land \alpha(t) \geq 0))$
  
  ... the expansion of $\alpha$ has no last non-negative rational
  
  and

- $\forall s (\alpha(s) \leq 0 \rightarrow \exists t (t > s \land \alpha(t) \leq 0))$
  
  ... the expansion of $\alpha$ has no last non-positive rational.
Definition: A real $\alpha$ is persistent if

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  ...the expansion of $\alpha$ has no last non-positive rational.

Theorem: ($\text{RCA}_0$) If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of persistent reals, then there is a set $I \subset \mathbb{N}$ such that for all $i$, $i \in I$ implies $\alpha_i \leq 0$ and $i \notin I$ implies $\alpha_i \geq 0$.

Theorem: ($\text{E-HA}^\omega$) If $\alpha$ is a persistent real, then $\alpha \geq 0$ or $\alpha \leq 0$.

Moral: Reverse math can assist in formulating constructive results.
Bibliography


Obligatory snapshots...
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