The strength of the polarized Ramsey's theorem

Damir D. Dzhafarov University of Chicago

Jeffry L. Hirst* Appalachian State University

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`www.mathsci.appstate.edu/~jlh`

Ramsey's theorem and a variant

Ramsey's theorem [RT^n_k]: If $f : [\mathbb{N}]^n \to k$, then we can find a $c < k$ and an infinite set H such that
$f(\{x_1, \ldots, x_n\}) = c$ for every $\{x_1, \ldots, x_n\} \in [H]^n$.

Polarized Ramsey's theorem [PT^n_k]: If $f : [\mathbb{N}]^n \to k$, then we can find a $c < k$ and infinite sets $H_1, \ldots, H_n$ such that $f(\{x_1, \ldots, x_n\}) = c$ for all $(x_1, \ldots, x_n) \in H_1 \times \cdots \times H_n$ with distinct components.

Increasing Polarized Ramsey's thm [IPT^n_k]: If $f : [\mathbb{N}]^n \to k$, then we can find a $c < k$ and infinite sets $H_1, \ldots, H_n$ such that $f(\{x_1, \ldots, x_n\}) = c$ for every
$(x_1, \ldots, x_n) \in H_1 \times \cdots \times H_n$ with $x_1 < \cdots < x_n$.

PT^n_k appears to be weak

If H is homogeneous for RT^n_k then $H \times \cdots \times H$ is homogeneous for PT^n_k.

$$\mathsf{RCA}_0 \vdash \forall n \forall k (\mathsf{RT}^n_k \to \mathsf{PT}^n_k)$$

Homogeneous sets for PT^n_k seem to contain less information than homogeneous sets for RT^n_k

For example, define $f : [\mathbb{N}]^2 \to 2$ by $f(x, y) = 0$ iff $x \equiv y \mod 2$. Every RT-homogeneous set for f must be 0 on all pairs. However, $H_1 = \{\text{evens}\}$, $H_2 = \{\text{odds}\}$ is a PT-homogeneous sequence with $f(x_1, x_2) = 1$ for all $(x_1, x_2) \in H_1 \times H_2$.

Initial question

Jim Schmerl asked: Is PT^n_k actually weaker than RT^n_k?

In particular, is PT provable in ACA_0?

Shorthand: PT abbreviates $\forall n \forall k \mathsf{PT}^n_k$

Short answer: For $n \geq 3$, PT^n_k is very similar to RT^n_k.

In particular, $\mathsf{ACA}_0 \nvdash \mathsf{PT}$.

Reverse mathematics of PT

Theorem: For every $n \geq 3$ and $k \geq 2$, RCA_0 proves

$$\mathsf{ACA}_0 \leftrightarrow \mathsf{RT}^n_k \leftrightarrow \mathsf{PT}^n_k \leftrightarrow \mathsf{IPT}^n_k$$

Comments:
A proof of $\mathsf{ACA}_0 \leftrightarrow \mathsf{RT}^n_k$ can be found in Simpson [6].
Proving $\mathsf{IPT}^3_2 \rightarrow \mathsf{ACA}_0$ is similar to the reversal for RT^3_2.

Theorem: RCA_0 proves

$$\mathsf{ACA}_0' \leftrightarrow \mathsf{RT} \leftrightarrow \mathsf{PT} \leftrightarrow \mathsf{IPT}$$

Comments:
$\mathsf{ACA}_0' \leftrightarrow \mathsf{RT}$ appears in J. Mileti's thesis [5].
$\mathsf{ACA}_0' = \mathsf{ACA}_0 + \forall n(\text{the } n^{th} \text{ jump exists})$.

Some computability theory

Fix $n, k \geq 2$.

Every computable $f : [\mathbb{N}]^n \to k$ has a Π^0_k definable PT-homogeneous sequence.

(Immediate from Jockusch [4].)

There is a computable $f : [\mathbb{N}]^n \to k$ with no Σ^0_n definable PT-homogeneous sequence.

(Adaptation of Jockusch [4].)

Pairs

Shorthand: RT^2 abbreviates $\forall k \mathsf{RT}^2_k$

Theorem: $\mathsf{RCA}_0 \vdash \mathsf{RT}^2 \leftrightarrow \mathsf{PT}^2$

Proof uses results on polarized Ramsey's theorem for stable colorings, and applies theorems of Cholak, Jockusch, and Slaman [1] and Hirschfeldt and Shore [3].

Question: Does $\mathsf{RCA}_0 \vdash \mathsf{IPT}^2 \rightarrow \mathsf{PT}^2$?

Question: How does this connect with the weak Ramsey principles of François Dorais?

Stable pairs

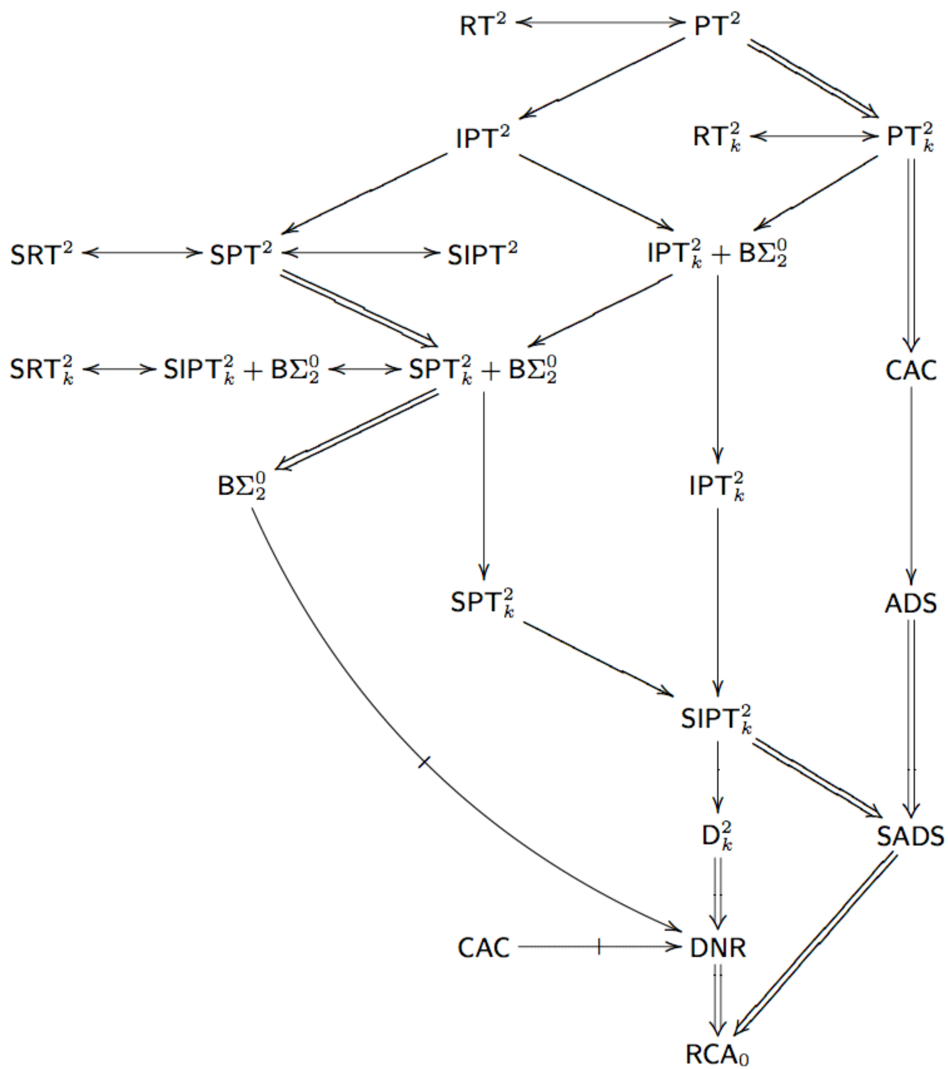
$f : [\mathbb{N}]^2 \to k$ is *stable* is $\forall m \lim_n f(m,n)$ exists.

SRT is RT restricted to stable colorings.

Theorem: $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2 \leftrightarrow \mathsf{SPT}^2 \leftrightarrow \mathsf{SIPT}^2$

Comment: For stable colorings, it is not so hard to generate an RT-homogeneous set from a IPT-homogeneous sequence, provided we can use the pigeonhole principle.

Question: Does $\mathsf{RCA}_0 \vdash \mathsf{SIPT}^2 \to \mathsf{IPT}^2$?



Results contributed by: Cholak, Dzhafarov, Hirschfeldt, Hirst, Jockusch, Kjos-Hanssen, Lempp, Slaman, and Shore

References

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