Reverse Mathematics and Persistent Reals

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These slides are available at: www.mathsci.appstate.edu/~jlh

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Motto: Dichotomy is not constructive.

A result familiar to constructivists:

**Theorem:** $(\mathsf{E-HA}_\omega^\omega + \mathsf{QF-AC}^{0,0})$ The following are equivalent:

1. **LLPO (Lesser limited principle of omniscience)** If $f : \mathbb{N} \to \{0, 1\}$ is a function that takes the value 1 at most once, then either $\forall n(f(2n) = 0)$ or $\forall n(f(2n + 1) = 0)$. 
2. If $\alpha$ is a real number, then $\alpha \geq 0$ or $\alpha \leq 0$.

Consequently, neither of these statements are provable in $\mathsf{E-HA}_\omega^\omega + \mathsf{AC}$.

**Exegesis:**

- $\mathsf{E-HA}_\omega^\omega + \mathsf{QF-AC}^{0,0}$ is a weak fragment of analysis based on intuitionistic predicate calculus.
- A real number is coded by a rapidly converging Cauchy sequence of rationals.
- If $\alpha > 0$, there is a witness. $\alpha \leq 0$ means $\neg(\alpha > 0)$. 
Motto: Dichotomy is computable, but...

**Theorem:** \((\text{RCA}_0)\) If \(\alpha\) is a real number, then \(\alpha \geq 0\) or \(\alpha \leq 0\).

\(\text{RCA}_0\) is a weak fragment of classical analysis, specifically, ordered semi-ring axioms plus induction for \(\Sigma^0_1\) formulas plus computable comprehension.

... but not uniformly computable.

**Theorem:** \((\text{RCA}_0)\) The following are equivalent:

1. \(\text{WKL}_0\) (Infinite 0–1 trees have infinite paths.)
2. If \(\langle \alpha_i \rangle_{i \in \mathbb{N}}\) is a sequence of reals, then there is a set \(I \subset \mathbb{N}\) such that for all \(i, i \in I\) implies \(\alpha_i \geq 0\) and \(i \notin I\) implies \(\alpha_i \leq 0\).
Ideas from the reversal

It suffices to use the statement about sequences of reals to find a separating set for the ranges of injections with disjoint ranges.
Ideas from the reversal

It suffices to use the statement about sequences of reals to find a separating set for the ranges of injections with disjoint ranges.

Suppose the injections look like this:

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(n)</td>
<td>4</td>
<td>9</td>
<td>5</td>
<td>8</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>g(n)</td>
<td>3</td>
<td>2</td>
<td>7</td>
<td>6</td>
<td>10</td>
<td>...</td>
</tr>
</tbody>
</table>

Then build these reals:

\[
\alpha_0 = \langle 0, 0, 0, 0, 0, \ldots \rangle
\]
\[
\alpha_1 = \langle 0, 0, 0, 0, 2^{-4}, 2^{-4}, 2^{-4}, 2^{-4}, \ldots \rangle
\]
\[
\alpha_2 = \langle 0, -2^{-1}, -2^{-1}, -2^{-1}, -2^{-1}, \ldots \rangle
\]

If \( I \) contains indices of non-negative reals and includes all positive reals, then \( I \) contains \( \{n \mid \alpha_n > 0\} \) and avoids \( \{n \mid \alpha_n < 0\} \), and so \( \text{range}(f) \subset I \) and \( \text{range}(g) \subset I^c \).
Since RCA$_0$ proves that sequential dichotomy implies WKL$_0$, RCA$_0$ cannot prove sequential dichotomy.

By a result of Hirst and Mummert [3], since RCA$_0$ cannot prove sequential dichotomy, E-HA$^\omega + AC + IP_{ef}^\omega$ does not prove dichotomy.

(AC is a choice scheme and IP$^\omega_{ef}$ is an independence of premise scheme for $\exists$-free formulas.)

The result from [3] is not a biconditional, but a *computable restriction* of sequential dichotomy can indicate a candidate for a *constructive* restriction of dichotomy.
Definition: A real $\alpha$ is persistent if

- $\forall s (\alpha(s) \geq 0 \rightarrow \exists t (t > s \land \alpha(t) \geq 0))$
  
  ... the expansion of $\alpha$ has no last non-negative rational and

- $\forall s (\alpha(s) \leq 0 \rightarrow \exists t (t > s \land \alpha(t) \leq 0))$
  
  ... the expansion of $\alpha$ has no last non-positive rational.

Theorem: ($\text{RCA}_0$)

If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of persistent reals, then there is a set $I \subset \mathbb{N}$ such that for all $i$, $i \in I$ implies $\alpha_i \leq 0$ and $i / \in I$ implies $\alpha_i \geq 0$.

Theorem: ($\hat{\text{E}} \text{-HA}_{\omega}$ $\downarrow$)

If $\alpha$ is a persistent real, then $\alpha \geq 0$ or $\alpha \leq 0$.

Moral: Reverse math can assist in formulating constructive results.
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Theorem: (RCA$_0$) If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of persistent reals, then there is a set $I \subset \mathbb{N}$ such that for all $i$, $i \in I$ implies $\alpha_i \leq 0$ and $i \notin I$ implies $\alpha_i \geq 0$.

Theorem: (\hat{E}-HA$^\omega$) If $\alpha$ is a persistent real, then $\alpha \geq 0$ or $\alpha \leq 0$.

Moral: Reverse math can assist in formulating constructive results.
Variations on persistence

**Definition:** A real $\alpha$ is $k$-persistent if

- $\forall s > k \ (\alpha(s) \geq 0 \rightarrow \exists t (t > s \land \alpha(t) \geq 0))$, and
- $\forall s > k \ (\alpha(s) \leq 0 \rightarrow \exists t (t > s \land \alpha(t) \leq 0))$.

**Definition:** $h$ is a *modulus of persistence* for $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ if for every $i$, $\alpha_i$ is $h(i)$-persistent.

**Theorem:** $(\text{RCA}_0)$ $\text{ACA}_0$ (arithmetical comprehension) is equivalent to “every sequence of reals has a modulus of persistence.”

**Theorem:** $(\text{RCA}_0)$ The following are equivalent:

1. $\text{WKL}_0$.
2. Every sequence of reals is component-wise equal to some sequence of 0-persistent reals.
3. Every sequence of reals is component-wise equal to a sequence that has a modulus of persistence.
Indices of minima

**Theorem:** [2] (RCA$_0$) The following are equivalent:

1. WKL$_0$.
2. For every sequence of reals $\langle \alpha_i \rangle_{i \in \mathbb{N}}$, there is a function $m: \mathbb{N} \to \mathbb{N}$ such that for each $n$, $\alpha_{m(n)} = \min\{\alpha_0, \ldots, \alpha_n\}$.

**Theorem:** (RCA$_0$) Fix $k$. If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence such that every initial segment is pairwise $k$-persistent, then there is a function $m$ such that for each $n$, $\alpha_{m(n)} = \min\{\alpha_0, \ldots, \alpha_n\}$.

**Theorem:** (E-HA$^\omega_1$ + QF-AC$^{0,0}$) Fix $k$. Every finite sequence of pairwise relatively $k$-persistent reals has a minimum.
Bibliography


