Two combinatorial proofs
and some related questions

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Proof 1: Hindman’s theorem implies ACA₀

**Hindman’s theorem (HT)** [6] If \( f : \mathbb{N} \to k \) then there is a color \( j \) and an infinite set \( X \subset \mathbb{N} \) such that whenever \( F \subset X \) is a finite set, \( f(\sum F) = j \).

The following form of Hindman’s theorem is provable equivalent over RCA₀ [1]:

**(FUT)** If \( f : \mathbb{N}^{<\mathbb{N}} \to k \) then there is a color \( j \) and an infinite increasing sequence of finite sets, \( X_0 < X_1 < X_2 < \ldots \) such that whenever \( F \subset \mathbb{N} \) is a finite set, \( f(\bigcup_{i \in F} X_i) = j \).

\( X_0 < X_1 < X_2 < \ldots \) means
\[
\max(X_0) < \min(X_1) < \max(X_1) < \min(X_2) < \ldots
\]
**Theorem:** Over RCA₀, FUT (and hence HT) implies ACA₀.  
(Theorem 2.2 of Blass, Hirst, and Simpson [1])

Ideas from the proof:

- Use FUT to prove that the range of an injection \( g \) exists.

- Given \( g \), define the coloring \( f : \mathbb{N}^\mathbb{N} \rightarrow 2 \).

- Apply FUT to \( f \) and verify that the range of \( g \) can be calculated from any monochromatic sequence.
Suppose $g : \mathbb{N} \rightarrow \mathbb{N}$ is an injection.

Given a finite set $X \subset \mathbb{N}$, define the very short gaps of $X$.

- Suppose $X$ is $x_0 < x_1 < x_2 < \cdots < x_n$.
- Say that $(x_i, x_{i+1})$ is a very short gap of $X$ if $x_i \cap g[x_{i+1}] \neq x_i \cap g[x_n]$.

Example:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(n)$</td>
<td>3</td>
<td>4</td>
<td>2012</td>
<td>2</td>
<td>6</td>
<td>7</td>
<td>0</td>
<td>8</td>
<td>9</td>
<td>1</td>
</tr>
</tbody>
</table>

$X$ is \{ 2, 5, 6, 8 \}

The range of $g$ on $(5, 8]$ contains an element less than 2, so (2, 5) is a very short gap.

The range of $g$ on (6, 8] contains no element less than 5, so (5, 6) is not a very short gap.
Suppose $g : \mathbb{N} \to \mathbb{N}$ is an injection.

Given a finite set $X \subset \mathbb{N}$, define the very short gaps of $X$.

- Suppose $X$ is $x_0 < x_1 < x_2 < \cdots < x_n$.
- Say that $(x_i, x_{i+1})$ is a very short gap of $X$ if
  \[ x_i \cap g[x_{i+1}] \neq x_i \cap g[x_n] \]
- Let $\text{VSG}(X)$ be the cardinality of the set of very short gaps of $X$.
- Define $f(X) = \text{VSG}(X) \mod 2$. ($f$ is a parity coloring.)

Apply FUT to find $S$, an increasing sequence of finite sets $X_0 < X_1 < X_2 < \ldots$ such that $f$ takes the same value on every finite union.
Short gaps vs. very short gaps

$(x_i, x_{i+1})$ is a short gap if the range of $f$ on $(x_{i+1}, \infty)$ contains an element less than $x_i$.

Example revisited:

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</tr>
</tbody>
</table>

$X$ is $\{2, 5, 6, 8\}$

$(2, 5)$ is a very short gap.

$(5, 6)$ is not a very short gap. $(6, 8)$ is not a very short gap.

Because $g(9) = 1$, $(2, 5)$, $(5, 6)$ and $(6, 8)$ are short gaps.

$(6, 10)$ might or might not be a short gap of $\{2, 6, 10\}$. 
SG vs. VSG

\((x_i, x_{i+1})\) is a short gap if the range of \(f\) on \((x_{i+1}, \infty)\) contains an element less than \(x_i\).

Example revisited:

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<tr>
<th>(n)</th>
<th>0</th>
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<th>4</th>
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\(X\) is \(\{2, 5, 6, 8\}\)

\((2, 5)\) is a very short gap.

\((5, 6)\) is not a very short gap. \((6, 8)\) is not a very short gap.

Because \(g(9) = 1\), \((2, 5)\), \((5, 6)\) and \((6, 8)\) are short gaps.

\((6, 10)\) might or might not be a short gap of \(\{2, 6, 10\}\).

\(\text{SG}(X)\) is shorthand for the number of short gaps of \(X\).
The monochromatic set encodes information about the short gaps

Recall: \( f \) takes the same value on every finite union of elements of \( S = \langle X_0, X_1, \ldots \rangle \). (Parity of VSG.)

**Claim:** If \( F \) is a finite union of elements of \( S \), then \( SG(F) \) is even.
Proof: Fix \( F \). Pick \( n \) so big that no value less than \( \max(F) \) appears in the range of \( g \) on \((\min(X_n), \infty)\). Consider \( F \cup X_n \).

- The short gaps of \( F \) are also very short gaps of \( F \cup X_n \).
- The very short gaps of \( X_n \) are also very short gaps of \( F \cup X_n \).
- \((\max F, \min X_n)\) is not a very short gap \( F \cup X_n \).
- Summarizing: \( VSG(F \cup X_n) = SG(F) + VSG(X_n) \)

Since \( VSG(F \cup X_n) = VSG(X_n) \mod 2 \), \( SG(F) \) is even.
Claim: If $F$ is a finite union of elements of $S$ and $X \in S$ satisfies $F < X$, then $(\max F, \min X)$ is not a short gap.

Proof: Visualize $\text{F}(\max F, \min X)\text{X}$. $\text{SG}(F \cup X_n) = \text{SG}(F) + \text{SG}([\max F, \min X]) + \text{SG}(X)$ so $0 = 0 + \text{SG}([\max F, \min X]) + 0 \mod 2$.

Claim: The range of $g$ is computable from $S$.

Proof: $\exists t(g(t) = n)$ iff $\exists t < \min X_{n+1} (g(t) = n)$
Proof 2: $\text{RT}_2^2$ implies the Free Set Theorem for pairs

Free Set Theorem for pairs ($\text{FS}(2)$): If $f : [\mathbb{N}]^2 \to \mathbb{N}$ then there is an infinite set $X \subset \mathbb{N}$ such that for all $(i, j) \in [X]^2$, if $f(i, j) \in X$ then $f(i, j) = i$ or $f(i, j) = j$.

**Theorem:** $\text{RCA}_0$ plus Ramsey’s theorem for pairs and two colors ($\text{RT}_2^2$) proves $\text{FS}(2)$. (Appears in Cholak, Guisto, Hirst, and Jockusch [2].)

Proof: Suppose $f : [\mathbb{N}]^2 \to \mathbb{N}$ and assume $\text{RT}_2^2$. We can use $\text{RT}_5^2$, if we like.
Define $g : \mathbb{N}^2 \to 5$ by the formula $g(i, j) = \begin{cases} 
0 & \text{if } f(i, j) < i \\
1 & \text{if } f(i, j) = i \\
2 & \text{if } f(i, j) \in (i, j) \\
3 & \text{if } f(i, j) = j \\
4 & \text{if } j < f(i, j) \end{cases}$ and let $M$ be an infinite set that is monochromatic for $g$.

- $g([M]^2) = 1$ implies FS(2) for $M$, since $f(i, j)$ is always $i$.
- $g([M]^2) = 3$ implies FS(2) for $M$, since $f(i, j)$ is always $j$.
- If $g([M]^2) = 4$, define $N \subset M$ by setting:
  $n_0 = m_0$, $n_1 = m_1$, and $n_{i+1}$ is the least element of $M$ greater than $\max\{f(n_j, n_k) \mid 0 \leq j < k \leq i\}$.
  $N$ satisfies FS(2).
A more challenging case:

\[ g([M]^2) = 0. \]  (So \( f(i, j) < i \) for all \( i, j \in M. \))

For each \( i \) and \( j \) define the sequence \( \sigma_{ij} \) as follows:
• A more challenging case:

\[ g([M]^2) = 0. \] (So \( f(i, j) < i \) for all \( i, j \in M \).)

For each \( i \) and \( j \) define the sequence \( \sigma_{ij} \) as follows:

Continue as long as the sequence decreases.

Define

\[ h : [M]^2 \to 2 \] by letting \( h(i, j) \) be the parity of the length of \( \sigma_{ij} \).

Apply \( \text{RT}^2 \) and find an infinite monochromatic set \( X \) for \( h \).

If \( i, j, \) and \( f(i, j) \) are all in \( X \), then \( \sigma_{ij} = f(i, j) \sim \sigma_{f(i,j),j} \), contradicting that their lengths have the same parity. Thus, if \( i, j \in X \), then \( f(i, j) \notin X \), so \( X \) satisfies \( \text{FS}(2) \).
• The last case: $g([M]^2 = 2$. (So $i < f(i,j) < j$ for all $i,j \in M$.)

As in the previous case, for each $i$ and $j$ define a sequence:

\[ i \rightarrow \tau_{ij}(0) \rightarrow j \]
The last case: \( g([M]^2) = 2 \). (So \( i < f(i, j) < j \) for all \( i, j \in M \).) As in the previous case, for each \( i \) and \( j \) define a sequence:

\[
\begin{array}{c}
  i \\
  \tau_{ij}(0) \\
  \tau_{ij}(1) \\
  j \\
\end{array}
\]

Repeat as long as the sequence increases. (It's bounded above by \( j \).)

Let \( h(i, j) \) be the parity of the length of \( \tau_{ij} \), and argue as before.
Suppose \( f : \mathbb{N}^{<\mathbb{N}} \rightarrow r \). Say that

- \( f \) is stable-t if for every \( t \) there is a \( b \) such that \( f \) is constant on all sets containing \( t \) and meeting \((b, \infty)\).
- \( f \) is stable-c if for every \( t \) there is a \( b \) such that \( f \) is constant on all sets containing \( t \) and of size at least \( b \).

Conj: \( \text{RCA}_0 \) proves that FUT for stable-t colorings is equivalent to polarized Ramsey’s Theorem for pairs.

Conj: \( \text{ACA}_0 \) proves FUT for stable-c colorings.

Does stable-c FUT imply \( \text{ACA}_0 \)?

Is there a related version of COH?
Thin set for unions (TSU): Suppose $f : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$. Then there is an infinite increasing sequence of finite sets, $X_0 < X_1 < X_2 < \ldots$ such that $\{f(\bigcup_{i \in F} X_i) \mid F \in \mathbb{N}^{<\mathbb{N}}\} \neq \mathbb{N}$.

There is also a free set theorem for unions FSU (see [2]).

Exer: RCA_0 proves HT $\to$ TSU.

Does TSU imply anything?

Prop: Milliken's theorem for triples implies FSU. [2]

Does HT imply FSU?

Does Milliken's theorem for triples imply the thin set theorem for all $k$?

Does the polarized thin set theorem imply anything?

How about the polarized free set theorem?
Bibliography


