Mathematical Logic as a Bridge Course

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`www.mathsci.appstate.edu/~jlh`

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Association for Symbolic Logic Guidelines[1]

All post-secondary institutions should offer at least one introductory course which teaches the basic notions of logic. All students should be encouraged to take such a course. These courses should include the following:

- The informal notion of "logically correct argument."
- Informal strategies for producing logically correct arguments and counterexamples to fallacious arguments.
- The propositional calculus as an example of a formal language, formal proofs, and the formalization of natural language arguments.

- A discussion of the relationship of proof, truth, and counterexamples, including a discussion of the Soundness Theorem.

- The predicate calculus extension of propositional logic.

- At least an informal discussion of the Completeness Theorem, if time permits.

[1]From *Guidelines for Logic Education* of the ASL Committee on Logic and Education, published in the Bulletin of Symbolic Logic, **1** (1995) pages 4–7.

An Appalachian State University Model

Techniques of Proof (MAT 2110)

A sophomore level bridge course directed (primarily) at mathematics majors.

Overview:

Begin with formal systems.

Transition to informal proof – viewed as a shorthand for formal axiomatic techniques.

Topics:

- Propositional calculus $(P \rightarrow P)$
- Predicate calculus $(\forall x P(x) \rightarrow P(x))$
- Equality
- Formal arithmetic
  (especially induction)
- Other topics:
  Sequences and convergence, continuity, linear algebra, other algebraic structures (groups, rings, fields), set theory, graph theory, incompleteness of arithmetic

Propositional Calculus

A brief discussion of truth tables and tautologies is followed by an extended treatment of formal proof, using …

The proof system **L** (as in Mendelson)

Axioms: (all instances of the following)

Ax 1: $A \to (B \to A)$

Ax 2: $(A \to (B \to C)) \to$
$$((A \to B) \to (A \to C))$$

Ax 3: $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

Rule of Inference:

MP (modus ponens)

From $A \to B$ and A, deduce B.

Some theorems of L: (Proved formally)

$$A \to B,\ B \to C \vdash_L A \to C$$

$$\vdash_L (\neg B \to \neg A) \to (A \to B)$$

$$\vdash_L \neg\neg B \to B$$

Mathematical logic theorems:

(Applied and justified, but not proved)

Deduction Theorem:

If $P \vdash_L Q$ then $\vdash_L P \to Q$.

Soundness Theorem:

If $\vdash_L P$, then P is a tautology.

Completeness Theorem:

If P is a tautology then $\vdash_L P$.

Predicate Calculus

Topics include truth, models, logical validity, quantifiers, free and bound variables, and formal proofs in predicate calculus.

Mathematical logic theorems:

(Applied and justified, but not proved)

Deduction theorem

Completeness theorem

Soundness theorem

Strategies for quantifier manipulation

Typical theorems:

$\forall x \forall y A(x, y) \ \vdash \ \forall y \forall x A(x, y)$

$\exists x (A(x) \rightarrow B(x)) \ \vdash \ \forall x A(x) \rightarrow \exists x B(x)$

$\neg \exists x A(x) \vdash \forall x \neg A(x)$

Advantages

- Fosters understanding of $\rightarrow$, $\neg$, and quantifiers – as used in proofs.
- "Rules" are theorems – students own them.
- Emphasizes importance of definitions and assumptions in informal mathematics.
- Provides a framework for further study.