AN EXACT BIFURCATION DIAGRAM FOR A REACTION DIFFUSION EQUATION ARISING IN POPULATION DYNAMICS

J. GODDARD II, Q. MORRIS, S. ROBINSON AND R. SHIVAJI

Abstract. We analyze the positive solutions to

$$\begin{cases} -\Delta v = \lambda v(1-v); & x \in \Omega_0, \\ \frac{\partial v}{\partial \eta} + \gamma\sqrt{\lambda} v = 0; & x \in \partial\Omega_0, \end{cases}$$

where $\Omega_0 = (0,1)$ or is a bounded domain in $\mathbb{R}^n$; $n = 2,3$ with smooth boundary and $|\Omega_0| = 1$, and λ, γ are positive parameters. Such steady state equations arise in population dynamics encapsulating assumptions regarding the patch/matrix interfaces such as patch preference and movement behavior. In this paper, we will discuss the exact bifurcation diagram and stability properties for such a steady state model.

1. Introduction

Let $\Omega_0 = (0,1)$ or be a bounded domain in $\mathbb{R}^n$; $n = 2,3$ with smooth boundary $\partial\Omega_0$ and $|\Omega_0| = 1$. Let $\Omega = \{\ell x \mid x \in \Omega_0\}$, where ℓ is a positive parameter. We will consider a population that satisfies a logistic growth in the patch Ω. We will assume that the diffusion rate in Ω

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is D and that Ω is surrounded by a matrix Ω_M, where the diffusion rate is D_0 and the death rate is S_0, all three positive parameters. Further assuming $r > 0$ is the patch intrinsic growth rate and $\kappa > 0$ is a parameter encapsulating assumptions regarding the patch/matrix interface such as patch preference and movement behavior, the resulting model is (see [10], [11], & [4]):

$$\begin{cases} u_t = D\Delta u + ru(1 - \frac{u}{K}); & x \in \Omega, \\ D\frac{\partial u}{\partial \eta} + \frac{\sqrt{S_0 D_0}}{\kappa} u = 0; & x \in \partial\Omega, \end{cases} \tag{1}$$

with steady state equation

$$\begin{cases} -\Delta u = \frac{1}{D} ru(1 - \frac{u}{K}); & x \in \Omega, \\ D\frac{\partial u}{\partial \eta} + \frac{\sqrt{S_0 D_0}}{\kappa} u = 0; & x \in \partial\Omega, \end{cases} \tag{2}$$

or equivalently:

$$\begin{cases} -\Delta u = \frac{\ell^2}{D} ru(1 - \frac{u}{K}); & x \in \Omega_0, \\ \frac{\partial u}{\partial \eta} + \frac{\ell S^*}{D\kappa} u = 0; & x \in \partial\Omega_0, \end{cases} \tag{3}$$

where K is the carrying capacity, and $S^* = \sqrt{S_0 D_0}$.

In this paper, we will be interested in the case when the density is continuous at the interface, that is when $\kappa = 1$. This corresponds to the case when the organisms move between the patch and the matrix with equal probability. Moreover, the step sizes and movement probabilities from the random walk are equal in the patch and the matrix (see [10] & [4]). Now applying the change of variables $v = \frac{u}{K}$, $\lambda = \frac{r\ell^2}{D}$, and $\gamma = \frac{S^*}{\sqrt{rD}}$, (3) reduces to:

$$\begin{cases} -\Delta v = \lambda v(1-v); & x \in \Omega_0, \\ \frac{\partial v}{\partial \eta} + \gamma\sqrt{\lambda} v = 0; & x \in \partial\Omega_0. \end{cases} \tag{4}$$

In this paper, we study existence, non-existence, uniqueness results, and stability properties for (4). To precisely state our results, we first consider the eigenvalue problem

$$\begin{cases} -\Delta w = \lambda w; & x \in \Omega_0, \\ \frac{\partial w}{\partial \eta} + \gamma\sqrt{\lambda} w = 0; & x \in \partial\Omega_0. \end{cases} \tag{5}$$

It follows that (5) has a principal eigenvalue $\lambda_1(\gamma) > 0$ and has a corresponding eigenfunction $w > 0$ in $\overline{\Omega}_0$ (see Appendix). We establish:

Theorem 1.1. *Given any $\gamma > 0$,*

(a) *If $\lambda > \lambda_1(\gamma)$ then the trivial solution of (4) is unstable and there exists a unique positive solution v_λ to (4) which is globally asymptotically stable. Furthermore, $\|v_\lambda\|_\infty \to 0^+$ as $\lambda \to \lambda_1(\gamma)^+$ and $\|v_\lambda\|_\infty \to 1$ as $\lambda \to \infty$;*

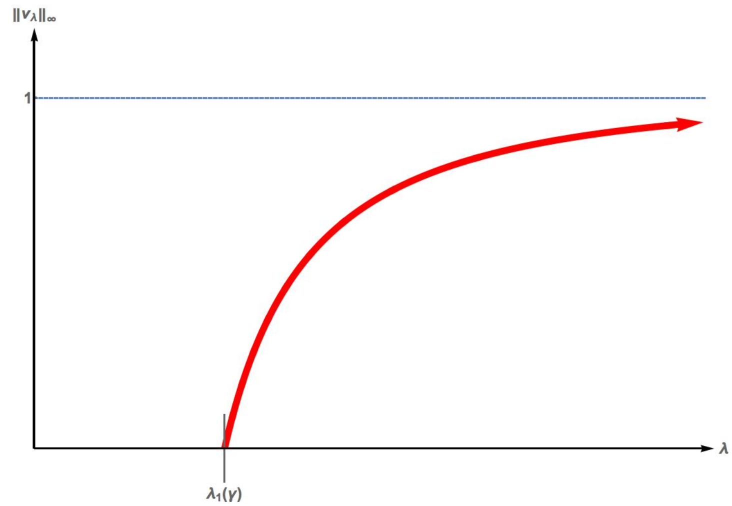

FIGURE 1. An illustration of the bifurcation curve for (4) as established in Theorem 1.1.

(b) *If* $\lambda \leq \lambda_1(\gamma)$ *then the trivial solution of* (4) *is globally asymptotically stable and there is no positive solution to* (4).

Note that $\lambda_1(\gamma) \to 0$ as $\gamma \to 0^+$.

We prove our results via the method of sub-super solutions and the principle of linearized stability. We provide the proof of Theorem 1.1 in Section 2. In Section 3, for the case $n = 1$, we provide an alternate proof of our results in the case $\Omega = (0, 1)$ via a quadrature method and discuss the evolution of the bifurcation curves as γ varies. In Section 4, we discuss biological implications of our results.

2. PROOF OF THEOREM 1.1

Let λ and γ be fixed and let σ_1 be the principal eigenvalue and $\phi > 0$ in $\overline{\Omega}_0$ be the corresponding eigenfunction such that $\|\phi\|_\infty = 1$ to the eigenvalue problem:

$$\begin{cases} -\Delta\phi - \lambda\phi = \sigma\phi; \quad x \in \Omega_0, \\ \frac{\partial\phi}{\partial\eta} + \gamma\sqrt{\lambda}\phi = 0; \quad x \in \partial\Omega_0. \end{cases} \tag{6}$$

Note that (6) is the linearization of (4) about the trivial solution. Clearly $\sigma_1 \geq 0$ when $\lambda \leq \lambda_1(\gamma)$ and $\sigma_1 < 0$ when $\lambda > \lambda_1(\gamma)$. Let $\lambda > \lambda_1(\gamma)$ and $\psi := \delta\phi$ for $\delta > 0$ to be chosen later. Then

$$-\Delta\psi - \lambda\psi(1-\psi) = \delta[\sigma_1 + \lambda\delta\phi]\phi; \;\; x \in \Omega_0,$$

and

$$\frac{\partial\psi}{\partial\eta} = \delta\frac{\partial\phi}{\partial\eta} = -\delta\gamma\sqrt{\lambda}\phi = -\gamma\sqrt{\lambda}\psi; \;\; x \in \partial\Omega_0.$$

Hence $\psi = \delta_1\phi$ with any $\delta_1 \in \left(0, -\frac{\sigma_1}{\lambda}\right)$ is a strict subsolution of (4) (since $\|\phi\|_\infty = 1$), and $\psi = \delta_2\phi$ with

$$\delta_2 = -\frac{\sigma_1}{\lambda\left[\min\limits_{\overline{\Omega}_0}\phi\right]}$$

is a supersolution of (4). Clearly $\delta_2 > \delta_1$, and hence by the method of sub-super solutions (see [8]), (4) has a positive solution $v_\lambda \in (\delta_1\phi, \delta_2\phi]$ for $\lambda > \lambda_1(\gamma)$. Note here that when $\lambda \to \lambda_1(\gamma)^+$, $[-\sigma_1] \to 0^+$ while $\min\limits_{\overline{\Omega}_0}\phi \not\to 0$ (see (5)). Thus, $\delta_1 \to 0^+$ and $\delta_2 \to 0^+$, and, in particular, $\|v_\lambda\|_\infty \to 0^+$ as $\lambda \to \lambda_1(\gamma)^+$.

Next, assume (4) has two positive solutions v_1 and v_2. Without loss of generality, we can assume v_2 is the maximal positive solution (since $w = 1$ is a global supersolution) and hence $v_2 \geq v_1$ in $\overline{\Omega}_0$. Supposing v_1 and v_2 are distinct, by integration by parts (Green's second identity), we obtain

$$\begin{aligned}
\int_{\Omega_0} [(\Delta v_2)v_1 - (\Delta v_1)v_2]\, dx &= \int_{\partial\Omega_0} \left[\left(\frac{\partial v_2}{\partial \eta}\right) v_1 - \left(\frac{\partial v_1}{\partial \eta}\right) v_2\right]\, ds \\
&= \int_{\partial\Omega_0} \left[\left(-\gamma\sqrt{\lambda} v_2\right) v_1 - \left(-\gamma\sqrt{\lambda} v_1\right) v_2\right]\, ds \\
&= 0,
\end{aligned}$$

while

$$\begin{aligned}
\int_{\Omega_0} [(\Delta v_2)v_1 - (\Delta v_1)v_2]\, dx &= \int_{\Omega_0} [(-\lambda v_2(1 - v_2))v_1 + (\lambda v_1(1 - v_1))v_2]\, dx \\
&= \int_{\Omega_0} \lambda v_1 v_2 (v_2 - v_1)\, dx \\
&> 0.
\end{aligned}$$

This is a contradiction, and hence $v_1 \equiv v_2$ and (4) has a unique positive solution v_λ for $\lambda > \lambda_1(\gamma)$.

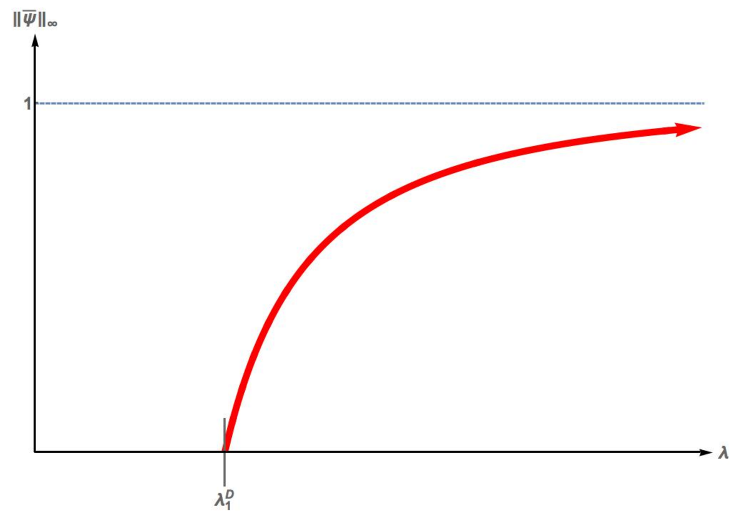

FIGURE 2. An illustration of the bifurcation curve for (7).

Note also that the Dirichlet boundary value problem:

$$\begin{cases} -\Delta\overline{\psi} = \lambda\overline{\psi}(1-\overline{\psi}); & x \in \Omega_0, \\ \overline{\psi} = 0; & x \in \partial\Omega_0, \end{cases} \tag{7}$$

has a unique positive solution $\overline{\psi}_\lambda$ for $\lambda > \lambda_1^D$ with $\|\overline{\psi}_\lambda\|_\infty < 1$ and $\|\overline{\psi}_\lambda\|_\infty \to 1^-$ as $\lambda \to \infty$, where $\lambda_1^D > 0$ is the principal eigenvalue of

$$\begin{cases} -\Delta w = \lambda w; & x \in \Omega_0, \\ w = 0; & x \in \partial\Omega_0. \end{cases} \tag{8}$$

Since $\frac{\partial\overline{\psi}}{\partial\eta} < 0$ in Ω_0, clearly $\overline{\psi}_\lambda$ is a subsolution to (4) for $\lambda >> 1$, and since $w = 1$ is a supersolution to (4) for $\lambda >> 1$, we must have $v_\lambda \in [\overline{\psi}_\lambda, 1]$ and hence $\|v_\lambda\|_\infty \to 1^-$ as $\lambda \to \infty$.

Also, when $\lambda \leq \lambda_1(\gamma)$ (which implies $\sigma_1 \geq 0$), if v is a positive solution of (4), by Green's second identity, we obtain

$$\begin{aligned}\int_{\Omega_0} [(\Delta v)\phi - (\Delta \phi)v]\ dx &= \int_{\partial\Omega_0} \left[\left(-\gamma\sqrt{\lambda}v\right)\phi - \left(-\gamma\sqrt{\lambda}\phi\right)v\right]\ ds \\ &= 0,\end{aligned}$$

while

$$\begin{aligned}\int_{\Omega_0} [(\Delta v)\phi - (\Delta \phi)v]\ dx &= \int_{\Omega_0} [(-\lambda v(1-v))\phi + \sigma_1\phi v + \lambda\phi v]\ dx \\ &= \int_{\Omega_0} \left[\lambda v^2\phi + \sigma_1\phi v\right]\ dx \\ &> 0.\end{aligned}$$

Hence, we have a contradiction and therefore (4) has no positive solutions when $\lambda \leq \lambda_1(\gamma)$.

The principle of linearized stability (see [13], for example) immediately gives that the trivial solution of (4) (as a steady state of (1)) is asymptotically stable whenever $\lambda \leq \lambda_1(\gamma)$ (which implies $\sigma_1 \geq 0$) and unstable whenever $\lambda > \lambda_1(\gamma)$ (which implies $\sigma_1 < 0$), where σ_1 is the principle eigenvalue of the linearized problem (6) associated with the trivial solution of (4). In the case that $\lambda > \lambda_1(\gamma)$, we have already shown the existence of a unique positive solution of (4), v_λ. Recall that we have $\psi = \delta_1\phi$ is a strict subsolution for all $\delta_1 \in \left(0, -\frac{\sigma_1}{\lambda}\right)$ and $Z \equiv M$ is a strict supersolution for all $M > 1$. This implies that $\phi < v_\lambda < Z$ and ϕ can be made arbitrarily small and Z can be made arbitrarily large. This fact combining with a result such as Theorem 5.6.7 of [13] immediately shows that v_λ is globally asymptotically stable. In the case that $\lambda \leq \lambda_1(\gamma)$, the stability of the trivial solution is global due to the nonexistence of a positive solution of (4). Hence, Theorem 1.1 is proven.

3. One-dimensional Problem

In the case $\Omega_0 = (0, 1)$, equation (4) reduces to the two-point boundary value problem

$$\begin{cases} -v'' = \lambda v(1-v); \quad x \in (0,1), \\ v'(0) = \gamma\sqrt{\lambda}v(0), \\ v'(1) = -\gamma\sqrt{\lambda}v(1). \end{cases} \tag{9}$$

From Theorem 1.1, (9) has a unique positive solution when $\lambda > \lambda_1(\gamma)$ and no positive solution when $\lambda < \lambda_1(\gamma)$, where $\lambda_1(\gamma)$ is the principle

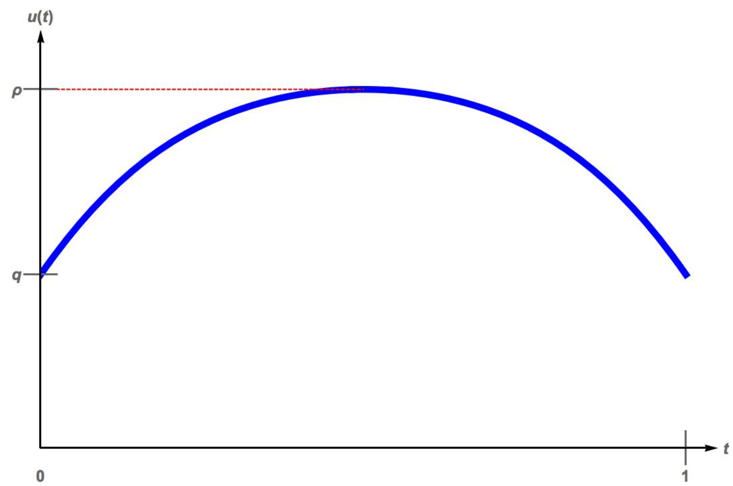

FIGURE 3. Shape of positive solutions to (10).

eigenvalue of

$$\begin{cases} -v'' = \lambda v; \quad x \in (0,1), \\ v'(0) = \gamma\sqrt{\lambda}v(0), \\ v'(1) = -\gamma\sqrt{\lambda}v(1). \end{cases} \tag{10}$$

A straightforward calculation will show that $\lambda_1(\gamma) = 4\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{\gamma}\right)\right)^2$. Note that as $\gamma \to 0^+$, $\lambda_1(\gamma) \to 0$ and as $\gamma \to \infty$, $\lambda_1(\gamma) \to \pi^2 = \lambda_1^D$.

We now use the quadrature method introduced by Laetsch in [9] and further extended in [1], [5], [6], [7], and [12]. Suppose u is a positive solution with $u(\frac{1}{2}) = \rho$ (say) and $u(0) = q$ (say). Note that since (9) is autonomous, the solution must be symmetric about $t = \frac{1}{2}$ and take the form shown in Figure 3.

Multiplying the differential equation in (9) by u' and integrating yields

$$u'(t) = \sqrt{2\lambda(F(\rho) - F(u(t)))}; \quad t \in \left[0, \frac{1}{2}\right]$$

where $F(z) = \int_0^z s(1-s)\,dt$. Further integration yields

$$\int_q^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}; \quad t \in \left[0, \frac{1}{2}\right],$$

and hence, setting $t \to \frac{1}{2}$, we obtain:

$$\sqrt{\lambda} = \sqrt{2}\left(\int_q^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}\right).$$

Now the boundary conditions require that ρ and q satisfy:

$$F(\rho) = \frac{2F(q) + \gamma^2 q^2}{2}. \tag{11}$$

Note that given $\rho \in (0,1)$, there exists a unique $q = q(\rho) \in (0, \rho)$ satisfying (11), and we can show that

$$G(\rho) = \sqrt{2}\int_{q(\rho)}^{\rho} \frac{ds}{F(\rho) - F(s)}$$

is well defined and continuous on $(0,1)$.

Further, given $\rho \in (0,1)$, for λ satisfying

$$\sqrt{\lambda} = G(\rho) = \sqrt{2}\int_{q(\rho)}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}}. \tag{12}$$

(9) has a positive solution of the form given in Figure 3 defined by:

$$\int_{q(\rho)}^{u(t)} \frac{ds}{\sqrt{F(\rho)-F(s)}} = \sqrt{2\lambda}t; \quad t \in \left[0, \frac{1}{2}\right).$$

Hence (12) describes the bifurcation diagram for positive solutions of (9). Using Mathematica computation, we provide below this bifurcation diagram for several values of γ. In particular, we illustrate the evolution of the bifurcation diagram as $\gamma \to 0^+$ and $\gamma \to \infty$ in Figures 4 and 5, respectively. Note that when $\gamma \to 0^+$, we approach the Neumann boundary condition case, and when $\gamma \to \infty$, we approach the Dirichlet Boundary condition case.

4. Biological Implications of Our Results

These model results give important predictions on population persistence at the patch level based solely on demographic parameters, e.g. patch diffusion rate and intrinsic growth rate, as well as matrix diffusion rate and death rate. We note that our assumption of $\kappa = 1$ or Continuous Density at the patch/matrix interface supposes that organisms do not detect the change between the patch and matrix. Thus,

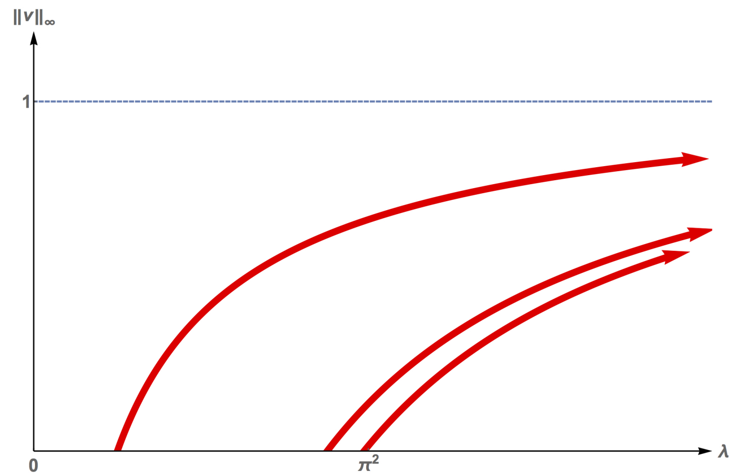

FIGURE 4. Bifurcation curves for (9) as $\gamma \to \infty$ generated in Mathematica. The curves correspond (from left to right) to $\gamma = 1$, $\gamma = 10$, and $\gamma = 100$. Note that as $\gamma \to \infty$, the bifurcation curves approach π^2, which is the first eigenvalue of the Dirichlet problem.

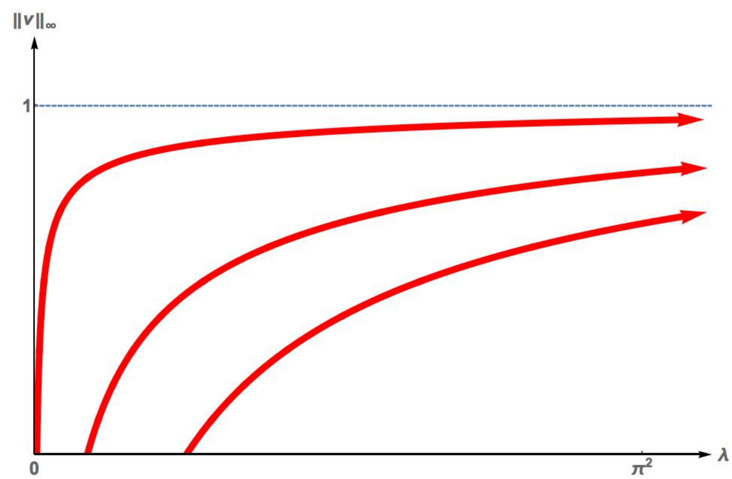

FIGURE 5. Bifurcation curves for (9) as $\gamma \to 0^+$ generated in Mathematica. The curves correspond (from left to right) to $\gamma = .1$, $\gamma = .5$, and $\gamma = 1$. Note that as $\gamma \to 0^+$, the bifurcation curves approach 0, which is the first eigenvalue of the Neumann problem.

they freely cross the boundary of the patch having a probability of remaining in the patch of 50% and do not adjust their movement behavior in the matrix. The principal eigenvalue, σ_1, of (6) plays a crucial role in determining the dynamics of the model. In fact, it represents the fastest possible growth rate for the linear growth model corresponding to (1) (see [3] or [2]).

As indicated in Theorem 1.1 (and the proof therein), when $\lambda \leq \lambda_1(\gamma)$ we have that $\sigma_1 \geq 0$ and the only nonnegative steady state of (1) is the trivial one, $u \equiv 0$. In this case, the model predicts extinction for any nonnegative initial population density profile. In fact, loses due to mortality in the matrix outpace the reproductive rate in the patch. Thus, the theoretical organism cannot colonize the patch and any remnant population in the patch will become extinct. However, when $\lambda > \lambda_1(\gamma)$ we have that $\sigma_1 < 0$ and (1) admits a unique steady state that is positive in $\overline{\Omega}$, such that all positive initial population density profiles will propagate to this steady state over time. In this case, the global nature of the stability of the positive steady state gives a fairly strong notion of persistence of the species. The patch is large enough in this case to shield a sufficient proportion of the population from mortality induced by the hostile matrix. This prediction leads to a formula for minimum patch size of the population given as $\ell^*(\gamma) =$

$\sqrt{\frac{D}{r}\lambda_1(\gamma)}$. Note that this formula can be numerically estimated and depends upon parameters in the patch (diffusion rate and intrinsic growth rate), parameters in the matrix via γ (diffusion rate and death rate), and the geometry of the patch Ω_0.

This notion of a minimum patch size agrees with the well known notion of a minimum core area (in the case of $n = 2$) requirement. Note that $\lambda_1(\gamma)$ can be viewed as a quantification of the loss of the population due to interactions with the hostile matrix where γ encapsulates parameters regarding the hostile matrix. Also, it is easy to see that $\lambda_1(\gamma) \to \lambda_1^D$ as $\gamma \to \infty$ and this reveals an important model prediction of the existence of a maximum possible effect of population loss due to the hostile matrix. Patches with a lethal matrix can still be guaranteed a prediction of persistence as long as the patch size is larger than $\sqrt{\frac{D}{r}\lambda_1^D}$, where the maximum effect of the lethal matrix on the population is quantified in λ_1^D. This minimum patch size approaches infinity if either 1) the patch diffusion rate is arbitrarily large, since a large diffusion rate ensures that a very high proportion of the population will encounter loss at the patch/matrix interface, or 2) the intrinsic growth rate is arbitrarily small, which for a fixed patch diffusion rate will imply that the population is not able to recover the loss associated with interaction with hostile matrix.

Appendix A. Some Results for the Eigenvalue Problem (5)

Consider the multiparameter eigenvalue problem

$$\begin{cases} -\Delta z = \lambda z; & x \in \Omega_0, \\ \frac{\partial z}{\partial \eta} = \mu z; & x \in \partial\Omega_0. \end{cases} \tag{13}$$

We recall the following result from [14].

Theorem A.1. *The first eigencurve $\lambda_1(\mu) \subset \mathbb{R}^2$ is Lipschitz continuous, strictly decreasing, and concave. Furthermore, $\lambda_1(0) = 0$, and the eigenfunction associated with any point on $\lambda_1(\mu)$ is strictly positive.*

Considering the eigenvalue problem (5), let $\mu = -\gamma\sqrt{\lambda}$. Define $A, B \subset \mathbb{R}^2$ by $A := \{(\mu, \lambda) \mid \lambda = \lambda_1(\mu)\}$ and $B := \{(\mu, \lambda) \mid \mu = -\gamma\sqrt{\lambda}\}$. Since A is concave and B is convex, it follows from Theorem A.1 that $A \cap B$ is a single point, say (μ_1, λ_1), with $\mu_1 = -\gamma\sqrt{\lambda_1}$. Therefore, (5) has a principal eigenvalue $\lambda_1(\gamma) > 0$ with corresponding eigenfunction $w > 0$ in Ω_0.

We now wish to show that $\lim_{\gamma\to\infty} \lambda_1(\gamma) = \lambda_1^D$. For any $\mu \in \mathbb{R}$, we may characterize $\lambda_1(\mu)$ by

$$\lambda_1(\mu) = \min_{u \in H^1(\Omega_0)\setminus\{0\}} \frac{\int_{\Omega_0} |\nabla u|^2 \, dx - \mu \int_{\partial\Omega_0} u^2 \, ds}{\int_{\Omega_0} u^2 \, dx}. \tag{14}$$

Let λ_1^D be the principle eigenvalue of (8) with corresponding eigenfunction ϕ_1^D chosen such that $\int_{\Omega_0} \phi_1^D = 1$. Testing (14) with $u = 1$ and $u = \phi_1^D$ shows that

$$\lambda_1(\mu) \leq -\mu \frac{|\partial\Omega_0|}{|\Omega_0|}$$

and

$$\lambda_1(\mu) \leq \lambda_1^D,$$

respectively. Taking a sequence $\mu_n \to -\infty$ such that the corresponding eigenfunctions u_n, without loss of generality, satisfy $\int_{\Omega_0} u_n \, dx = 1$, we observe that

$$\lambda_1(\mu_n) = \int_{\Omega_0} |\nabla u_n|^2 \, dx - \mu_n \int_{\partial\Omega_0} u_n^2 \, ds.$$

Since $\mu_n < 0$, we have $0 = \lambda_1(0) < \lambda_1(\mu_n) < \lambda_1^D$.

By Theorem A.1, $\lim_{\mu \to -\infty} \lambda_1(\mu) = \lambda_1(-\infty) \leq \lambda_1^D$ for some $\lambda_1(-\infty) \in \mathbb{R}$. Without loss of generality, we may assume $-\mu_n \int_{\partial\Omega_0} u_n^2 \, ds \to \alpha \geq 0$, and thus $\int_{\partial\Omega_0} u_n^2 \to 0$.

Since $\{u_n\}$ is bounded in $H^1(\Omega_0)$, we may select a subsequence so that $u_n \rightharpoonup u$ in $H^1(\Omega_0)$, $u_n \to u$ in $L^2(\Omega_0)$ and in $L^2(\partial\Omega_0)$. It follows that $\int_{\Omega_0} u^2 \, dx = 1$ and $\int_{\partial\Omega_0} u^2 \, ds = 0$, and hence $u \in H_0^1(\Omega_0)$.

By the weak lower semicontinuity of $\int_{\Omega_0} |\nabla u|^2 \, dx$, we get that,

$$\int_{\Omega_0} |\nabla u|^2 \, dx + \alpha \leq \liminf_{n \to \infty} \left(\int_{\Omega_0} |\nabla u_n| \, dx - \mu_n \int_{\partial \Omega_0} u_n^2 \, ds \right) = \lambda_1(-\infty) \leq \lambda_1^D.$$

But by Poincare's Inequality, we have $\lambda_1^D \leq \int_{\Omega_0} |\nabla u|^2 \, dx$, and hence we must have $\alpha = 0$ and $\lambda_1(-\infty) = \lambda_1^D$. Furthermore, $\int_{\Omega_0} |\nabla u|^2 \, dx = \lambda_1^D$, and thus, without loss of generality, $u = \phi_1^D$. Moreover, $\lim_{n \to \infty} \int_{\Omega_0} |\nabla u_n|^2 \, dx = \int_{\Omega_0} |\nabla u|^2 \, dx$, and hence $u_n \to u = \phi_1^D$ in $H^1(\Omega)$.

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