Research Statement

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My research is in the area of nonlinear partial differential equations, in particular, positive steady states of reaction diffusion equations, which have wide applications in areas such as population dynamics, combustion theory, and chemical reactor theory. In general, I consider problems of the form

$$\begin{cases} -\Delta_p u = \lambda f(u), & x \in \Omega, \\ Bu = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\lambda > 0$ is a real parameter, $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z)$ with $p > 1$, $f : [0, \infty) \to \mathbb{R}$ is a continuous function, $\Omega \subset \mathbb{R}^N$, and B is a generic, possibly nonlinear, operator to be specified later.

Pure PDE Research

Of particular interest in my pure PDE research is the case where f satisfies the additional conditions,

(F1) $f(0) < 0$, (semipositone), and

(F2) there exists $A, B \in (0, \infty)$ and $q \in (p-1, \infty)$ such that for $s > 0$ sufficiently large, $As^q \leq f(s) \leq Bs^q$ (which implies that f is p-superlinear at infinity).

The semipositone nature of f poses a significant challenge in establishing positivity of a solution (see [6], [16]). There is a rich history of such problems, particularly in the case $Bu = u$ (Dirichlet boundary conditions) and Ω is a bounded domain in $\mathbb{R}^N$. See [1–5, 7, 10–12, 14, 15, 17] for results regarding superlinear, semipositone problems where Ω is a general bounded domain. My research to date focuses on establishing the existence of a positive solution to problems of this type on exterior domains (see Figure 1).

We have considered two problems on the domain exterior to a ball, with the results published in [QM1] and [QM2], respectively. In particular, we consider consider the problems

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u), & x \in \Omega_e, \\ u = 0, & |x| = r_0, \\ u \to 0, & |x| \to \infty, \end{cases} \tag{2}$$

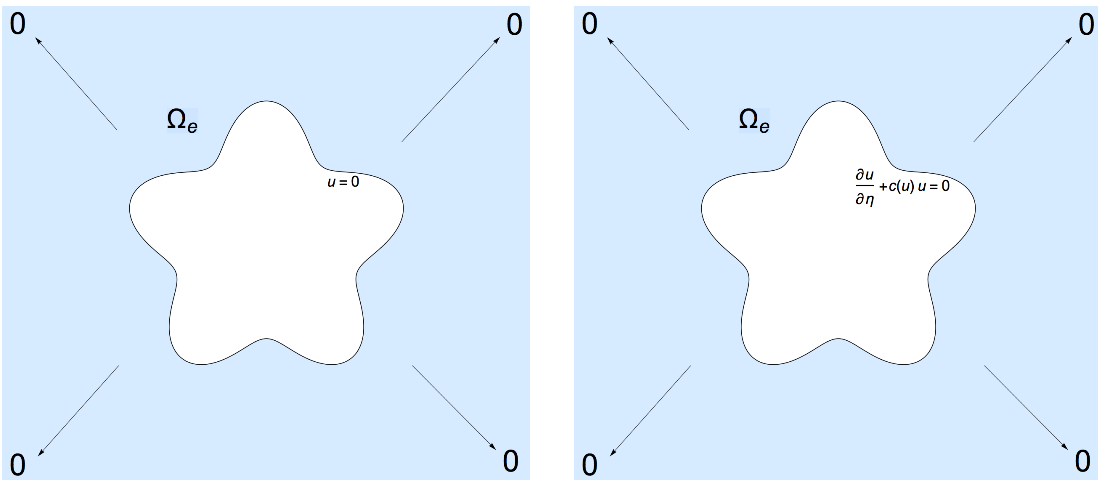
and

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u), & x \in \Omega_e, \\ \frac{\partial u}{\partial \eta} + \tilde{c}(u) u = 0, & |x| = r_0, \\ u \to 0, & |x| \to \infty, \end{cases} \tag{3}$$

where,

$$\Omega_e = \left\{ x \in \mathbb{R}^N \mid |x| > r_0, r_0 > 0, N > p \right\},$$

$K \in C([r_0, \infty), (0, \infty))$ satisfies $K(r) \leq \frac{1}{r^{N+\mu}}$; $\mu > 0$ for $r >> 1$, $\frac{\partial}{\partial \eta}$ is the outward normal derivative, and $\tilde{c} \in C([0, \infty), (0, \infty))$. In [QM1], we consider the semilinear cases of (2) and (3), $p = 2$, where $\Delta_2 = \Delta$ is the usual Laplace operator. In [QM2], we treat the quasilinear cases of (2) and (3), $p > 1$, where Δ_p is the p-Laplace operator, and in [QM3], we establish a numerical algorithm for the generation of exact bifurcation curves. Current work aims to establish the uniqueness of these solutions, as well as prove the existence of solutions on general exterior domains.



(a) Dirichlet boundary condition as in (2). (b) Nonlinear boundary condition as in (3).

Figure 1: Exterior domains with different boundary conditions.

I have also studied existence results for problems of the form

$$\begin{cases} -u'' = au^+ - bu^- + g(u), \\ u(0) = u(2\pi), \\ u'(0) = u'(2\pi), \end{cases} \tag{4}$$

where $(a, b) \in \mathbb{R}^2$, $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$, and $g : \mathbb{R} \to \mathbb{R}$ is a bounded, continuous function. In [QM4], we prove existence of a solution in the resonance case subject to a generalized orthogonality condition as well as existence of a solution in the nonresonance case with no further restrictions, a result later generalized to the PDE case in [13]. I am currently working with N. Mavinga on a related PDE problem, which includes the functions u^+ and u^- in both the differential equation and the boundary condition.

Mathematical Ecology Research

As part of an interdisciplinary study "Mathematical and Experimental Analysis of Ecological Models: Patches, Landscapes and Conditional Dispersal on the Boundary," I have treated the existence and multiplicity of solutions to the equation

$$\begin{cases} -\Delta u = \lambda u(1-u), & x \in \Omega, \\ \frac{\partial u}{\partial \eta} + \gamma\sqrt{\lambda}(u-A)^2 u = 0, & x \in \partial\Omega, \end{cases} \tag{5}$$

where $\lambda > 0$ is a real parameter, and $\gamma > 0$ and $0 < A < 1$ are constants related to the quality of the surrounding environment and dispersal rate on the boundary, respectively. Solutions to (5) are steady states of a reaction-diffusion equation which models a population with U-shaped density dependent dispersal on the boundary (i.e., high rates of dispersal when population size is very high or very low and low rates of dispersal for intermediate population sizes; see Figure 2).While the case of positive density dependent dispersal has been well researched (see [8], [9]), the case of negative density dependence has been treated in few cases and the case of U-shaped density dependence has not been treated at all. In [QM5] and [QM6], we show existence, multiplicity, and uniqueness results for certain ranges of the parameter λ, and explore how these results depend on the constant γ. Our theoretical results will be compared to experimental results of collaborators Jim Cronin and Rachel Harman of LSU (see Figure 3).

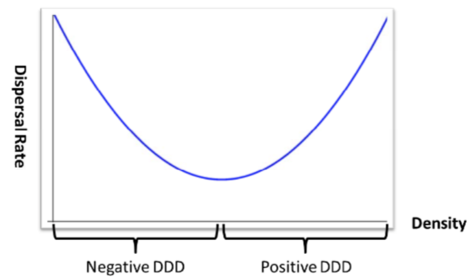


Figure 2

(a)

(b)

(c)

Figure 3: Mathematical Ecology Research Group: (a) Full research group at Cameron Jetties in January 2016; (b) LSU ecology collaborators at the field site; (c) Full research group at UNCG in June 2016.

While my applied research to date has focused on mathematical ecology, my interests in mathematical modeling and using tools of nonlinear analysis to analyze mathematical models are broader than these specific examples. I am interested in collaboration with faculty members from any number of departments from biology to economics who have an interest in PDE modeling.

Future Research Focus

The work summarized above, in both pure and applied contexts, provides many avenues for extension. In expanding my pure PDE research, I plan to address the following problems:

1. **Existence on non-radial domains:** While the work in [QM1] and [QM2] proves the existence of a positive radial solution on the exterior of a radial domain, I would like to extend these results to the non-radial cases by again employing variational methods. In particular, one may first consider solutions on the exterior of a ball without assuming a radial structure of the solution. Beyond that, one may also consider solutions on the exterior of a non-radial domain. While a mountain pass solution is tractable in the correct variational setting, showing the positivity of the solution in these cases would be more challenging.

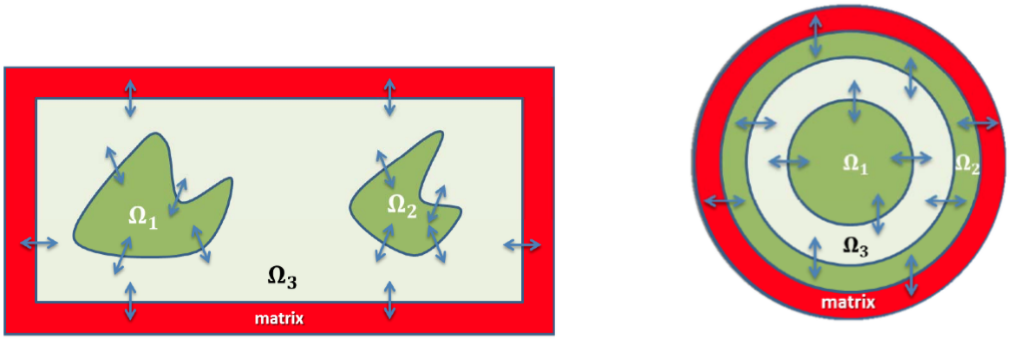
2. **Uniqueness:** In addition to extending the existence results, I would also like to establish uniqueness of the radial solutions obtained in [QM1] and [QM2]. While uniqueness results for such superlinear, semipositone problems are often difficult to come by, I will attempt to employ bifurcation theory and implicit function theorem arguments to establish this result.

3. **Fučík Spectrum:** Combining my interests in nonlinear boundary conditions from [QM1] and [QM2] with my interests in resonance and nonresonance problems from [QM4], I am working to establish the existence of solutions to an analogue of (4) when the boundary conditions are nonlinear.

In the mathematical ecology field, I plan to continue work with the same research group to extend the results of [QM6] to address the following questions:

1. **Multiple patches separated by matrix:** I aim to extend the model framework in order to study true landscape level dynamics of a population via a reaction diffusion system for a landscape model consisting of two patches, Ω_1 and Ω_2, separated by an intermediate matrix Ω_3 that has the ability to maintain a small population (see Figure 4a). In particular, by extending the landscape model to the case of a two-dimensional landscape consisting of a disk Ω_1 and an annulus Ω_2, separated by an intermediate matrix Ω_3 that has the ability to maintain a small population. The annulus Ω_2 is then surrounded by a tunable hostile matrix (see Figure 4b). The framework will be sufficiently flexible as to allow different dynamics in each of the patches and the intermediate matrix, as well as conditional dispersal at each of the patch boundaries. I will study the structure of radially symmetric positive steady state solutions of the reaction diffusion based landscape model in concert with stability of the steady states. The reaction terms and dispersal behaviors in [QM6] would be considered for study in this extension.

2. **Competing species model:** Again adapting earlier models, I aim to develop a competing species reaction diffusion system to model competition mediated dispersal. In this case, two



(a) Generic multiple patch model. (b) Radially symmetric model.

Figure 4: Multiple patch models

competing organisms occupy the same patch $\Omega = \{\ell x \mid x \in \Omega_0\}$ surrounded by a tunable hostile matrix. The main assumptions of the model are: 1) potential competitors have no effect on within-patch diffusion patterns, and 2) the emigration rate increases and/or the shape of the density-dispersal curve changes in the presence of an interspecific competitor. In other words, an unbiased random walk is assumed throughout the patch, while on the patch boundary organisms bias their movement in that their dispersal (emigration) from the patch is dependent on the density of the other organisms. I will then study the existence, nonexistence, uniqueness, multiplicity, and stability of positive steady state solutions of the model as the patch size ℓ and matrix hostility S^* are both varied. Predictions of the persistence of the population will then be related to the biologically meaningful parameters providing crucial insight into the effects of habitat fragmentation and predation on the patch level dynamics of the population.

Publications

[QM1] R. Dhanya, Q. Morris, and R. Shivaji. Existence of positive radial solutions for superlinear, semipositone problems on the exterior of a ball. *J. Math. Anal. Appl.*, 434(2):1533–1548, 2016.

[QM2] Q. Morris, R. Shivaji, and I. Sim. Existence of a positive radial solution for superlinear, semipositone p-Laplacian problem on the exterior of a ball. *To appear in Proc. Roy. Soc. Edinburgh Sect. A*, 2017.

[QM3] J. Goddard II, Q. Morris, B. Son, and R. Shivaji. Bifurcation curves for some singular and nonsingular problems with nonlinear boundary conditions. *To appear in Electron. J. Differential Equations*, 2018.

[QM4] Q. Morris and S. Robinson. A Landesman-Lazer condition for the boundary value problem $-u'' = au^+ - bu^- + g(u)$ with periodic boundary conditions. In *Proceedings of the Ninth Mississippi State–UAB Conference on Differential Equations and Computational Simulations*, volume 20 of *Electron. J. Differ. Equ. Conf.*, pages 103–117, 2013.

[QM5] J. Goddard II, Q. Morris, S. Robinson, and R. Shivaji. An exact bifurcation diagram for a reaction diffusion equation arising in population dynamics. *Submitted to Commun. Pure Appl. Anal.*, 2017.

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