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CHAPTER 7

Mathematical description of operant behavior: an introduction

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1. **Introduction**

Behavior analysts are interested in describing functional relations between classes of independent variables and classes of dependent variables. To be most useful such descriptions should be precise, succinct, and applicable to a wide range of particular cases. In psychology, including behavior analysis, most scientific descriptions are couched in a modified version of everyday language (e.g., English). Familiar-sounding terms are redefined and new terms are introduced to form a technical vocabulary. Such languages may be satisfactory for describing general trends. But they are clumsy for describing precise relationships.

The language of mathematics, in contrast, encourages descriptions that are precise, succinct and general. An equation, for example, describes precisely and succinctly how the terms are related to each other. The term on the left usually specifies the dependent variable (i.e., some property of behavior). The classes of independent variables are included among the terms on the right. The other terms on the right, and the organization of the right-hand terms in the equation, specify how the independent and dependent variables are related. What an equation can say precisely in a few lines might take several paragraphs to say in English.

Sometimes people speak of mathematical description as a *mathematical model.* An equation can be viewed as a model of some empirical phenomenon in the sense that it is a construction intended to resemble the phenomenon in certain important respects but not in all respects. The relevant aspects of the concrete empirical events should change in the same way as the corresponding terms of the equation.

An equation that provides a good description of the relevant aspects of the relation between the independent and dependent variables can aid prediction and control. The effects of changing various independent variables can be explored quickly and easily in the model by changing the appropriate terms and solving the equation. The mathematical model (or mathematical description) will be useful to the extent that it does, in fact, predict well and efficiently over an appropriately wide range of circumstances.

2. Generating a mathematical model

If you are unfamiliar with mathematical description, the process of coming up with plausible equations to try can seem mysterious, and you may feel in awe of those who have been successful. But, in fact, the processes are not mysterious (Skinner, 1957, 1969; Marr, 1985, 1986; Langley et al., 1987). Historical and experimental evidence suggests that even the most spectacular achievements in mathematical description result from conventional problem-solving heuristics (Langley et al., 1987). Getting ideas about what classes of equations to try can come from analogies with other phenomena (i.e., metaphorical

extension), as illustrated in Section 2.1, and from patterns revealed in plots of data, as illustrated in Section 2.2.

2.1. Metaphorical extension

A common strategy for generating ideas for mathematical models is to conceptualize the behavioral phenomenon as analogous to some other phenomenon - behavioral or even physical - for which effective mathematical descriptions already have been developed. Then, with appropriate adjustments and translations, the mathematical description can be applied to the new case. This is a form of metaphorical extension (cf. Langley et al., 1987; Skinner, 1957, 1989). Some examples are described below.

2.1.1. Example 1. Stimulus sampling theory

Under many learning procedures, the probability of a response increases as a function of trials. But the size of the increase progressively decreases with each successive trial. Prior to 1950 most mathematical models of learning (e.g., Hull, 1943) were expressed in the form of *deterministic functions.* Each value of the independent variable implied a particular value of the dependent variable. The obvious moment-to-moment variability in performance was attributed to the effect of additional uncontrolled variables.

With *stochastic* or *probabilistic* models, in contrast, each value of the independent variable can imply a frequency distribution of values of the dependent variable. Thus, variability is an intrinsic implication of the model rather than something treated as an additional effect.

A novel contribution of Estes (1950, 1959) was to conceptualize the learning process as analogous to some prototypic probabilistic process. First, Estes took learning to be the development of stimulus-response relations. Then, following Guthrie (1935) and others, Estes imagined that the stimulus was composed of a large number of independent elements. On any conditioning trial, only some subset of the total number of elements are effectively noticed. Of those, some elements already may have been conditioned to evoke the response while others have not. The ones that have not been conditioned are conditioned on that trial in an all-or-none fashion. The tendency to respond is assumed to be ordinally related to the number of noticed elements that have been conditioned.

This conception makes learning analogous to an urn problem of the following sort. Imagine an urn containing 100 white balls (analogous to the total number of elements comprising the stimulus). On each trial we randomly draw 10 balls (analogous to the noticed subset). We then mark each unmarked ball in our sample with an X (analogous to a conditioning effect), and put the balls back before the next trial. With each successive trial, progressively more of the balls in the urn will be marked so that, on average, the sample will contain fewer and fewer unmarked balls. More formally:

$$\Delta M_i = S(N - M_{i-1})$$

where ΔM_i indicates the expected number of new balls marked on any particular trial (i); S refers to the proportion of the total number of balls comprising a sample; N indicates the number of balls in the urn; and M_{i-1} indicates the total number of marked balls at the end of the previous conditioning trial. ΔM_i will get smaller and smaller on each successive trial.

The mathematical description of these sorts of urn problems had been well worked out, at least for prototypic cases (e.g., Feller, 1950). The real key, then, was in conceptualizing aspects of learning as analogous to the urn problem. To the extent that there are analogous aspects, the mathematical descriptions developed for the urn problem should provide an effective description of interesting properties of learning provided the terms are appropriately translated. The history of *stimulus sampling theory* (Estes, 1959) and its descendants (e.g., Rescorla and Wagner, 1972) demonstrates that it was indeed productive.

2.1.2. Example 2: Herrnstein's response-rate function

The development of stimulus sampling theory illustrates metaphorical extension from physical phenomena. Herrnstein's (1970, 1974) development of a function relating the rate of a response to its rate of reinforcement illustrates metaphorical extension from one behavioral phenomenon to another.

Two facts about behavior were relevant. One was the effect of rate of reinforcement on the emission rate of a single response. Suppose, as Skinner (1938) did, that reinforcement increased and nonreinforcement decreased the emission rate of a response. If so, the average response rate under stable conditions of intermittent reinforcement should be a blend of these two effects, with higher reinforcement rates changing the blend in favor of higher response rates.

Catania and Reynolds (1968) attempted to determine this function precisely by examining the rate of responding generated by extended exposure to each of a large number of variable-interval (VI) schedules of food reinforcement. The subjects were food-deprived pigeons. The results were that response rate increased as reinforcement rate increased, but the function was curved. It rose less and less steeply as reinforcement rate increased - that is, it was an increasing, negatively accelerated function (see Section 2.2).

The second relevant fact came from choice procedures where each response is reinforced on a different VI schedule (concurrent VI VI schedules; see Part 1, Ch. 6). Herrnstein (1961) and others (cf. Catania, 1966) discovered that a simple linear equation describes the relation between the proportion of responses and the proportion of reinforcers:

$$\frac{B_1}{B_1 + B_2} = \frac{R_1}{R_1 + R_2} \tag{1}$$

where B indicates responses; R indicates reinforcers; and the subscripts identify the particular alternative.

The question was whether these two facts about behavior- the curvilinear relation between response and reinforcement rates and the linear relation between response and reinforcer proportions in a choice procedure - were related. Several theorists (e.g., Bush and Mosteller, 1955; Estes, 1959) had tried to bridge choice and single response procedures by conceptualizing the single response procedure as analogous to a choice procedure but where the alternative responses are not measured directly. In this tradition, Herrnstein (1970) conceptualized the single response procedure, such as studied by Catania and Reynolds, as a concurrent schedule: the measured response plus all the other unmeasured responses (turning, grooming, exploring and so forth). Further, he imagined that these other responses are reinforced by unscheduled, perhaps intrinsic, reinforcers. Next, he assumed that the simple matching principle holds for choice, whether or not the responses and reinforcers are measured. These possibilities can be expressed formally. Assuming matching among all the responses and reinforcers:

$$\frac{B_1}{B_1 + B_2 + ... + B_N} = \frac{R_1}{R_1 + R_2 + ... + R_N}$$

or,

$$\frac{B_1}{\sum B_i} = \frac{R_1}{R_1 + R_o}$$

where R_o represents the rate of all other reinforcers and where responses and reinforcers from all sources are measured in the same units as the measured response and the scheduled reinforcer. Then to obtain the number of responses during the session time (response rate), we can simply multiply by the denominator on the left, giving:

$$B_1 = \frac{\left(\sum B_i\right) R_1}{R_1 + R_o}$$

Finally, Herrnstein (1974) assumed that the total amount of behavior is constant for a given session duration. Simply put, the subject is always doing something, even if sleeping, so that the total of all the 'somethings' per unit time does not change, only their distribution. If so, the equation can be rewritten, finally, as:

$$B_1 = \frac{kR_1}{R_1 + R_o} \tag{2}$$

where *k* represents the total rate of all behavior, expressed in units of the target response. Or, in other words, *k* represents the asymptotic rate of the target response in the limiting case where all the reinforcement is assigned to the target response. If the sum of all alternative reinforcement, R_o, *is* assumed to be constant, then Herrnstein's equation is a hyperbolic, which is a rising, negatively accelerated function. (The hyperbolic and other function types will be described more fully in Section 2.2.) In words, the equation says that response rate is an increasing, negatively accelerated function of its own reinforcement rate and a decreasing function of the rate of alternative reinforcement.

The hyperbolic function fit Catania and Reynolds' and a wide range of other data well, provided appropriate values were chosen for the two constants, *k* and *Ro* (deVilliers and Herrnstein, 1976; Davison and McCarthy, 1988). It also describes well some data obtained in applied settings (McDowell, 1982; Myerson and Hale, 1984; Bradshaw and Szabadi, 1988), and so applies with some generality.

A word of caution is in order, however. The fact that an equation of particular form describes a set of data well does not mean that the assumptions that gave rise to the equation are supported. It turns out, for example, that an equation of the same hyperbolic form as Herrnstein's can be derived from very different sets of assumptions (e.g., Catania, 1973; Staddon, 1977; Killeen, 1979; McDowell, 1980; and Equation 6). And the constants in the equation accordingly have different interpretations under these different derivations. As will be discussed below, the constants, which are derived from a data set, are really dependent variables even though they may be interpreted as representing the effects of unmeasured independent variables.

The point of these two examples (Estes and Herrnstein) is to emphasize that developing an effective mathematical description is neither a magical process nor a mechanical process of feeding a set of data into some curve-fitting routine and getting the 'correct' function out. Often the most important part is to conceptualize the phenomenon of interest in a way that reveals analogies with other phenomena for which mathematical descriptions have already been worked out. Then the derivation for the present case may be straightforward, even though the implications might turn out to be surprising.

There are many other examples within the operant tradition of this same sort of metaphorical extension including: McDowell's application of linear system theory from engineering (e.g., McDowell, 1987; McDowell and Wixted, 1988); applications of models developed for economic and regulatory systems (e.g., Baum, 1973, 1981; Lea, 1978; Staddon and Motheral, 1978; Rachlin et al., 1978; Timberlake, 1984; Collier et al., 1986; Hursh et al., 1988; Staddon and Ettinger, 1989); and applications derived from extensions of models of time-based probabilistic processes (e.g., Gibbon, 1979; Luce, 1986; Killeen and Fetterman, 1988; McGill, 1963). Most models within operant theory are static ones in the sense that they describe the asymptotic adjustment but not the time course of that adjustment. There are, however, many examples of dynamic and semi-dynamic models developed in other fields that might be applied by analogy (cf. Staddon, 1988, for a useful discussion of this class of models).

2.2. Familiar function forms in plots of data

Metaphorical extension, as just described, is one way to generate ideas about what equations to try. A second way, to be discussed in this section, is to plot a set of data and see whether the pattern suggests a familiar type of equation.

Sometimes the plot of points appears chaotic, perhaps indicating that the independent variable simply has no systematic effect on the aspect of behavior being measured or that influential variables are insufficiently controlled. But often an orderly relation can be discovered through further data exploration

- i.e., through plotting the data different ways (cf. Tukey, 1977). For example, an orderly relation might emerge if the independent or dependent variables are reexpressed.

If the relation appears orderly, one may be able to see in the plot of data certain familiar trends that narrow the range of possibilities. Does the trend rise from left to right (a *positive* function) or fall (a *negative* function)? If the trend remains positive or negative, it is *monotonic* (i.e., one limbed). The normal curve, in contrast, is *bitonic* (i.e., two limbed). If a function (either positive or negative) becomes progressively less steep, going from left to right, then it is *negatively accelerated.* If it becomes progressively steeper, then it is *positively accelerated.*

The particular shape of any function will vary, depending on exactly how the X- Axis and Y-axis variables are scaled, whether linear, logarithmic, and so forth. Which scaling is appropriate? Linear-scale expressions of the independent and dependent variables are most common. Such scales are linear transformations of common physical measurements. But it does not necessarily follow that such scaling is appropriate for behavioral relations. The familiar physical scales have evolved for effective description in physics. Different scales or complex transformations of conventional physical scales may yield simpler, more effective, and thus more natural descriptions of behavioral relations. Such descriptions are completely physical; but the scales are not the familiar ones from descriptions of physical phenomena (cf. Stevens, 1951; Gibson, 1966).

2.2.1. Obtaining data

The first step is to obtain enough data points to determine a functional relation between the independent and dependent variables over a sufficiently wide range. Although you may need to generate the critical data yourself, you may be able to find them in journal articles or book chapters. When procedures are similar across laboratories, data from different experiments and laboratories can be compared meaningfully. Data can be obtained from tabular presentations or from published graphs if you can enlarge them enough to read the values accurately (cf. Gibbon, 1977). You can prepare a transparency of the graph (e.g., by Xeroxing) that can be projected on a screen. Then adjust the size so that the scale values marked on the graph correspond to those on a standard grid that you have prepared. Next use the grid template to determine the values of the points from the projected image. The task goes quickly if one person reads the values and another records them.

2.2.2. Linear plots

Suppose the points are plotted on linear coordinates. If the points appear to scatter around a rising or falling straight line, the familiar linear form, $y = ax + b$, *is* immediately suggested. The *a is* the slope, or proportionality constant; it determines the steepness of the line. The *b is* the intercept constant; it indicates where the line will meet the Y-axis when the value of x is zero. If *a is* positive, the line will rise from left to right; if *a is* negative, the line will fall.

If, however, the trend of the points deviates systematically from a straight line, an equation other than a linear one is called for. There are an infinite number of such functions. But a relatively small number of monotonic functions are immediately suggested because they have been found effective for describing behavioral relations.

The upper-left panel of Fig. I shows several such functions plotted in linear coordinates. You can see how the direction and curvature differ depending on the type of function and on the values of the constants. Consider, for example, a *power function* of the form, $y = ax^b$ (Curves B and C in the top left panel). The exponent, *b,* determines the rate of acceleration in the function relating y to x. If *b is* greater than I (Curve B in Fig. 1), y is an increasing, positively accelerated function of x; if *b is* equal to 1, y is a linear function of x (zero intercept); and if *b is* less than I (Curve C in Fig. 1), y is an increasing, negatively accelerated function of x. Finally, a negative exponent is equivalent to raising the number to the appropriate power and then taking the reciprocal of the result. For example, X^{-2} is equivalent to $1/X^2$, which is a decreasing function.

For power functions, equal relative changes in the independent variable produce equal relative changes in the dependent variable. The power (i.e., *b)* sometimes is called the *sensitivity parameter* because its value determines what relative change in the dependent variable is produced by a given relative change in the independent variable. Power functions have proven useful for describing such things as the way stimulus and reinforcement differences between alternatives affect choice (e.g., Baum, 1974, 1979; Nevin, 1984; Davison and McCarthy, 1988; Logue, 1988; and Part 1, Ch. 6).

For the *exponential function* (Curves D and E in Fig. 1) the independent variable (x) operates as an exponent. If the exponent is positive (Curve D), y is an increasing, positively accelerated function of x. If the exponent is negative (Curve E), y is a decreasing, negatively accelerated function of x. Exponential functions have proven useful for describing certain growth and decay processes. For example, the negative exponential describes the discharge of an electrolytic capacitor, the distribution of interevent times generated by a random emitter, and the decay of radioactive substances. For the exponential, the amount gained or lost during a time interval of particular duration is a constant proportion of the amount

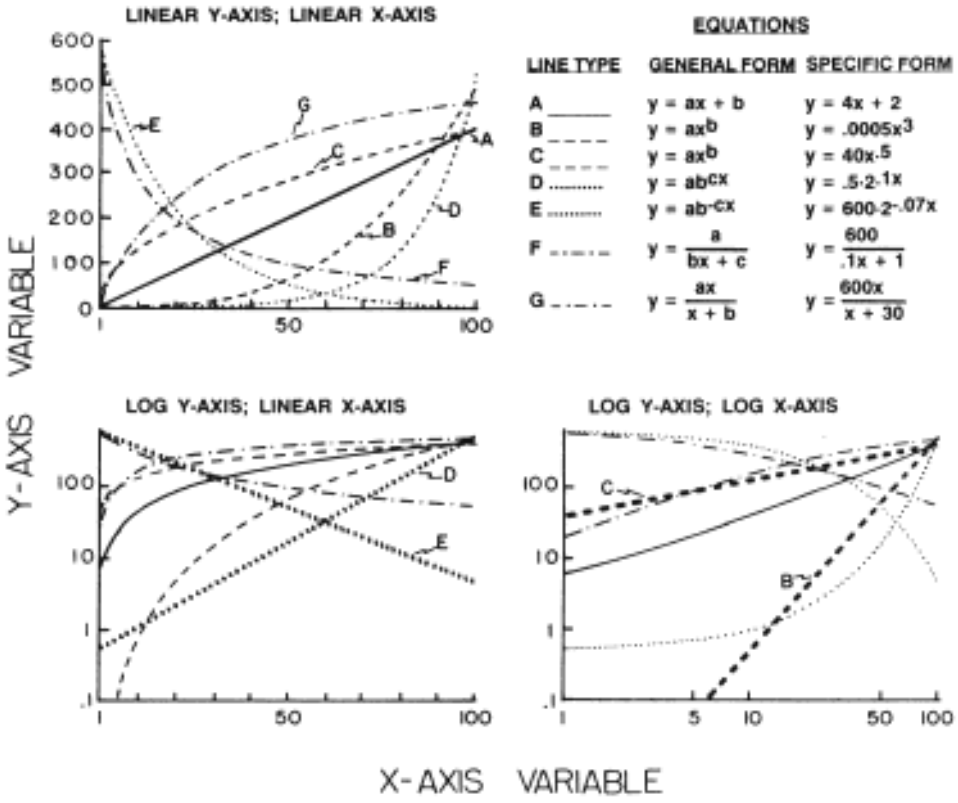

Fig. 1. Several common functions plotted in linear coordinates (top left panel), in semi-logarithmic coordinates (lower left panel), and in full logarithmic coordinates (lower right panel). The functions that are linear in each type of plot are shown by thickened lines. Curve A is a linear function; B and C are power functions; D and E are exponential; and F and G are hyperbolic.

at the beginning of the time interval. In the analysis of behavior the negative exponential has proven useful for describing such phenomena as the distribution of interresponse times (Blough, 1966; see Section 4.1.2), the decay of activity after the discontinuation of reinforcement (e.g., Killeen, 1975, 1979), and the deterioration of stimulus control as a function of the retention interval in delayed matching to sample (remembering) tasks (e.g., White and McKenzie, 1982; Watson and Blampied, 1989).

Herrnstein's equation relating response rate to reinforcement rate (Section 2.1.2) is a type of *hyperbolic function* where the X-axis variable appears both in the numerator and in the denominator (Curve G in Fig. 1). Here, y is an increasing, negatively accelerated function of x (see also deVilliers and Herrnstein, 1976, and McDowell and Wood, 1984). When the X-axis variable appears only in the denominator (Curve F in Fig. 1), y is a decreasing, negatively accelerated function of x. Hyperbolic functions like Curve F have proven useful for describing the effects of delayed reinforcement on choice (e.g., Rachlin

and Green, 1972; Ainslie, 1975, 1989; Herrnstein, 1981; Mazur, 1987; and Section 4.4.1; see also Gibbon, 1977).

It should be apparent from Fig. 1 (top left) that several different kinds of equations produce trends that appear quite similar. The differences may appear mainly at the extremes. Nonetheless, it might be important to decide which one provides the better description because the equations might imply quite different kinds of controlling relationships. The decision might determine how we conceptualize the phenomenon we are studying and what other phenomena we relate it to. The question, then, is which function provides the best description of a set of data.

2.2.3. Logarithmic plots

Because it is easier to detect systematic deviations from a straight than from a curved line, it is often helpful to find a transformation that converts a curved function into a linear function. When the dependent and independent variables are reexpressed that way, the points should scatter unsystematically around a straight line if, indeed, the function provides an adequate description. Systematic deviations are likely to be apparent by eye. Two of the curved functions in Fig. 1 (the power function and the exponential function) have familiar and easy linear reexpressions involving logarithms.

A logarithm is the exponent that some base number (usually either 10 or *e)* must be raised to in order to equal the target number. Logarithms based on *e (e* = 2.718...) are called natural logarithms and are often abbreviated *ln* (e.g., on calculators); those based on 10 are called common logarithms and are often abbreviated *log.* Thus, for example, ln(x)) would be read as the logarithm, base *e,* of the value specified by x. Although the numerical values of the two kinds of logarithms will differ, as long as one is consistent, it will not matter which type of logarithmic reexpression one uses. The logarithm of the base is 1.0; the logarithm of 1 is zero; the logarithm of a decimal is negative in sign; and the logarithm of 0 is undefined. Distances on the logarithmic scale correspond to ratios of the original numbers. To convert an equation to logarithmic form, exponents become multipliers and the logarithms of numbers that are multiplied in the original expression are added. For example, if $y = (10^2)(50)$, then log(y) = 2(log(10)) + log(50). For a power function of the form, $y = ax^b$:

log(y) = *b* log(x) + log(a)

which is linear. In order to emphasize its linear form, it might be helpful to express this relation in a slightly different way: let y' = log(y), a' = log(a) and x' = log(x). Then for the power function:

$y' = b\,x' +$ a'

Thus, the power function is linear when *both* the x and y variables have been converted to logarithms. And the slope in logarithmic coordinates is the exponent (b) in the power function (i.e., the sensitivity parameter).

The linearity of logarithmic reexpressions of power functions makes it relatively easy to see whether a power function provides a good description of the trend in a set of data. One can convert the X-axis and Y-axis values of each data point to its logarithm and then plot these logarithmic values. Alternatively, one can plot the original data values on double logarithmic graph paper - both axes scaled logarithmically - which is readily available with different numbers of logarithmic cycles. The logarithmic scaling on the graph paper is equivalent to transforming the coordinate values of the variables to their logarithms. Either way, if the trend of the points appears linear, a power-function description is suggested. The Generalized Matching Law (e.g., Baum, 1974; Davison and McCarthy, 1988; see also Part 1, Ch. 6) is a power function and is often expressed in its logarithmic form.

In the lower right panel of Fig. 1 both the X- and Y-axis values are scaled logarithmically (i.e., double logarithmic paper). The point is to show the shape of the various functions in double-logarithmic coordinates. Note that the power functions are linear, but the other function types are not. Thus, if the trend of the data is nonlinear in double logarithmic coordinates, another function type should be considered.

If the exponential function, $y = ae^{bX}$, is converted to its logarithmic form (base e), the e-term drops out because the logarithm of the base is 1.0, and so

$\ln(y) = b\,x + \ln(a)$, or

$y' = bx + a'$

Thus, the exponential function plots as a straight line when the Y-axis is scaled logarithmically and the X-axis is scaled linearly.

The lower left panel of Fig. I shows the several function types plotted in semi- logarithmic coordinates (logarithmic Y- axis, linear X-axis). Note that the exponential functions, but not the other functions, plot as straight lines.

It is important to recognize that many kinds of functions which are curved in linear coordinates are also curved in full-logarithmic and semi-logarithmic coordinates. In Fig. 1, for example, the hyperbolic functions are linear in none of the plots. This fact may be useful for distinguishing among different functions. For example, it would be hard to distinguish a power function (power less than 1.0) from a hyperbolic function if the data were plotted in linear coordinates (e.g., compare Curves C versus G in the upper left plot). But the two functions appear different in the double logarithmic plot (lower right panel).

The best way to become familiar with these various patterns is to plot these kinds of functions with real or made-up data. If you would like a more extended treatment of the basic function types and of logarithmic and other reexpressions, Lewis' (1960) Quantitative Methods in Psychology would be a good place to start.

Fig. 2 illustrates the effects of logarithmic reexpressions on two sets of behavioral data. The data in the top row show for one pigeon the effect of the retention interval on a measure of discrimination in a delayed matching-to-sample task (Pigeon T5, easy discrimination, Experiment 2 in White and McKenzie, 1982); see also Part 1, Ch. 8. The data in the lower row show for one pigeon the effect of changing an aspect of the temporal distribution of reinforcers in the terminal link of a chain schedule (see Section 4.2.1) on the rate of initiating the terminal link (Pigeon 3819 in Shull et al., 1990); see also Part 1, Ch. 3. For the left-most panels in each row, the points are plotted in linear coordinates. For the middle column, the dependent variable is expressed in logarithmic units (i.e., y'=ln(y)). And for the right-hand column both the independent and dependent variables are expressed in logarithmic units.

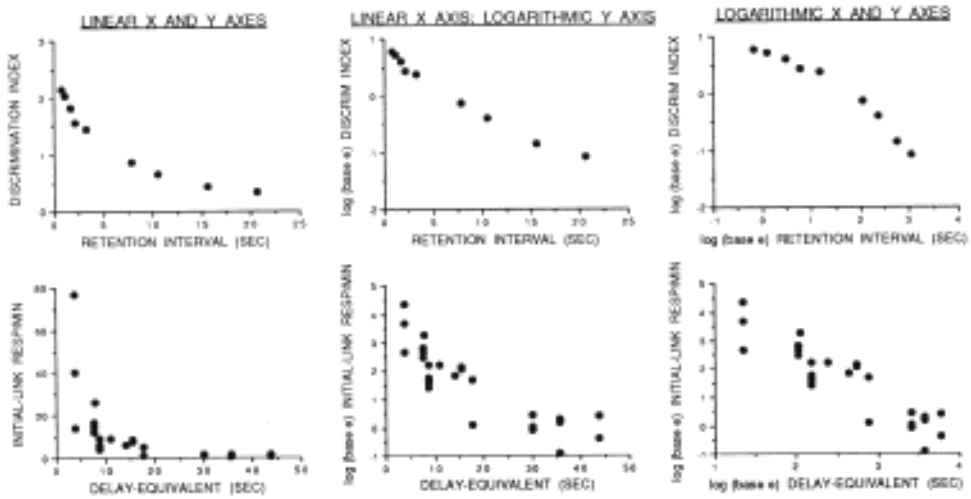

Fig. 2. Scatter plots showing the effects of logarithmic reexpression. The top row shows the values of a discrimination index from one pigeon (Pigeon T5) from Experiment 2, easy discrimination, reported by White and McKenzie (1982). The bottom row shows initial-link response rates from one pigeon (Pigeon 3819) trained under a chain schedule with FR I in the initial link and various numbers and delays of food reinforcers in the terminal link. (Adapted from Shull et al., 1990, with permission of the publisher.)

Again, the aim is to see if the trend in the data is linear in any of these plots because such linearity would indicate that a particular kind of function provides a good description of the trend. Conversely, systematic deviations from a straight- line trend would show that the function type that plots as a straight line does not provide a completely satisfactory description. A good way to detect curvature is to hold the graph horizontal to the ground at about chin level, with the right-hand end of the X-axis pointing toward your face. Then sight along the function. Holding the graph this way effectively shortens the X-axis relative to the Y-axis and exaggerates curvature.

Consider first the left-most panels in both rows (linear coordinates). Clearly, a linear function provides a poor description of the trends because the trends are bowed inward (i.e., decreasing, negatively accelerated). Consider next the data in the bottom row only. The trend is bowed inward in semi-logarithmic coordinates (middle panel) but is reasonably linear in full-logarithmic coordinates (right-hand panel). Thus, a negative power function, but not a negative exponential function, describes the trend well. Consider finally the data in the top row. Here the trend appears to deviate from a straight line in all three plots. Yet among the plots of points in the top row, the one in the middle (semi-logarithmic) appears most nearly linear, suggesting that a negative exponential offers the best description of the three.

There are different reasons for reexpressing data. The reason emphasized here is that reexpressions which produce a linear trend can help one decide how well a particular function type describes a given set of data. A different reason for reexpressing data is to make the data set better conform to the distributional requirements for interpreting certain statistical calculations and tests (e.g., they may equalize variances or make the distributions more nearly normal in form). Whether or not a certain reexpression will prove useful or appropriate will depend, in part, on the intended purpose of the reexpression.

Other functions, distributions, and mathematical techniques can become familiar and are readily suggested by certain kinds of patterns in the data sets to those who are experienced with them. Probability theory provides an especially rich source. Often there are conventional transformations to make evaluation easier.

2.3. Curve fitting

The next step is to fit the theoretical function to the set of data. Many text books are available on the topic of regression analysis and curve fitting (e.g., Mosteller and Tukey, 1977; Kleinbaum et al., 1988). The statistical issues can be complex and can have important implications for interpreting the results. Thus, consultation with a professional statistician is advisable.

2.3.1. Parameter estimation and interpretation

Normally a mathematical model is an equation with a term representing some aspect of behavior on the left and a set of interrelated terms on the right. The terms on the right are of two sorts. Some of the terms refer to concrete, manipulable events (i.e., independent variables). Others are *curve-fitting parameters* (also called *curve-fitting constants* or *free parameters).*

The values of the independent and dependent variables are known from the procedure and data. But the values of the other terms are not known. Parameter estimation, or curve-fitting, procedures find the values that the curve-fitting parameters must have for the equation to fit the data as well as it can, according to some criterion of 'best fit' (e.g., Restle and Greeno, 1970). Since the values of the curve-fitting constants are extracted from the data, they are best regarded as *dependent variables.*

Usually the curve-fitting constants are given additional meaning outside their role in the equation. That is, the constants may be given a name that is meaningful within some broader, perhaps nonmathematical, theoretical conception. Sometimes those names seem to imply that the terms refer to genuine independent variables. For example, the lower right-hand term in Herrnstein's hyperbolic equation, R_o, *is* described as referring to other reinforcement, including unscheduled reinforcement, in the situation (see Equation 2 above). But, in fact, R_o *is* operationally defined very differently from how scheduled

reinforcers are defined. The value of R_o is the result of a curve-fitting operation, not the result of measuring how frequently some event follows a response. Whether or not we wish to interpret that constant as analogous to rate of reinforcement (an independent variable) will depend on considerations apart from how well an equation of the particular form fits the data.

2.3.2. Linear regression and other techniques

The simplest and most familiar technique is fitting a straight line to a set of data by means of Pearson's product-moment method. This method gives the best fitting straight line based on the criterion of minimizing the sum of the squared deviations between the data points and the line. There are two curve-fitting constants, the slope and the intercept, that are determined from the data. The procedure can be applied to the original data or to the reexpressions intended to produce a linear form. Other techniques are available to fit functions to nonlinear functions. Sometimes these involve a sort of 'getting warm-getting cold' process, where the program tries different curve-fitting parameters, evaluates the fit, and then adjusts the parameters according to some rule to see if the fit is improved.

2.4. Means

2.4.1. Arithmetic, geometric and harmonic means

Average values (e.g., means) also can suggest the most useful forms of equations describing the relation between independent and dependent variables. Imagine a set of observations ordered along some scale. Think of the observations as equal weights placed at the appropriate places on a wooden ruler. The mean of that set is analogous to the balance point along the ruler. Now imagine that the ruler/scale is stretched at the low end and compressed at the high end. The weight at the low end would be moved farther from and the weights at the high end would be moved closer to the original balance point, and so the balance point would shift leftward - toward lower values. More generally, the balance point, or the mean, depends on the scale along which the observations are ordered. The mean will be analogous to the balance point only if it is calculated based on numbers expressed in units appropriate to the scale.

For example, logarithmic reexpressions (x'= log(x)) in effect stretch the low end and compress the high end of scales. Thus, the mean of the original numbers (i.e., the *arithmetic mean)* will be high relative to the balance point on the logarithmic scale. To find the balance point on the logarithmic scale, one would need to calculate the mean of the logarithmic reexpressions: convert each number to its logarithm, sum the logs, and divide by the number of observations. The anti-log of the mean of the logs is the *geometric mean* of the original set of numbers. For example, the geometric mean of 10 and 100 can be found by taking the anti-log of the mean of the logs: since the log of 10 is I and the log of 100 is 2, the mean of the logs is 0.5(1 +2)= 1.5, and so the geometric mean is the anti-log of 1.5 (i.e., 31.6). The *arithmetic mean,* in contrast, is 0.5(10 + 100) = 55. Similarly, reciprocal reexpressions (x'= 1/x) stretch the low end and compress the high end relative to the original scale. Thus, if the reciprocal reexpression is appropriate, then the mean of the reciprocals would be the appropriate description of the balance point. The reciprocal of the mean of the reciprocals is the *harmonic mean* of the original numbers (cf. Killeen, 1968). To calculate the harmonic mean of 10 and 100, first get the mean of the reciprocals: 0.5[(1/10) + (1 / 100)] = 0.5(0.1 +0.01) = 0.055. Then take the reciprocal of the mean of the reciprocals: (1/0.055)= 18.2.

The same logic would apply to any scale reexpression: convert the original numbers to units of the new scale, calculate the mean of these reexpressions, and finally convert the mean of the reexpressed values back to units of the original scale. This new mean will differ from the mean of the original numbers depending on the relation between the two scales.

Because the mean is a balance point, it is very sensitive to extreme values (i.e., to skew). Thus, if a distribution of data is skewed, the mean will be misleading if it is interpreted as representing the typical value (cf. Stevens, 1955). One kind of solution is to reexpress the data so that the distribution becomes more nearly symmetrical, and then calculate the mean based on the reexpressed set. Calculating means of ratio data illustrates this kind of problem and solution. Imagine a choice procedure where, let us assume, there really is no preference. Suppose for unknown reasons the subject distributes its responses between the left and right alternatives on a 2:1 ratio on one day and on a 1:2 ratio on the next. We can see first what happens if we calculate the arithmetic mean of these two ratios: 0.5(2+0.5)= 1.25. That mean

suggests, misleadingly, that choice favors the left alternative. The reason is that 2 is a greater linear distance from I than is 0.5 even though a 2:1 choice ratio seems intuitively equivalent to a 1:2 ratio. Equal distances on a logarithmic scale correspond to equal ratios on the original scale. Thus, the geometric mean seems appropriate for this problem: 0.5[ln(2)+ln(0.5)]=0.5[0.693+(-0.693)] = 0; and the anti-log of 0 is 1. And so the geometric mean suggests, appropriately, that neither alternative is favored over the other.

As just described, if you know the original numbers and the scale-reexpression rule, you can calculate the balance point (mean) for the distribution of reexpressed values. And then you can covert that mean to the corresponding value in the original scale. But often the scale-reexpression rule is precisely what you are trying to discover rather than what you know in advance. It may be possible to discover the reexpression rule by, in effect, working backward from the mean value. Suppose you know the original-scale values of the events that comprise a distribution. And suppose also that you can determine the balance point of the distribution. Then you can determine what transformation rule, applied to the original set of numbers, will result in the known mean value. More formally, let *M* represent the balance point of a distribution comprised of some number (N) of values (X_i). And let f represent the reexpression rule. (For the geometric mean, for example, the f specifies a logarithmic reexpression; for the harmonic mean, the f specifies a reciprocal reexpression.) Then

$$f(M) = (1/N)\left(\sum f(X_i)\right)$$

The analytic task, then, is to find the form off that satisfies the equation. The experimental task is to discover the balance point of the set of values, which sometimes can be accomplished with choice procedures (see Section 4.4.1 and Part 1, Ch. 6). The basic strategy is to determine what fixed value along a dimension is functionally equivalent to a distribution of values. The equivalent fixed value is the balance point of the distribution. For example, one might give subjects (e.g., pigeons) a choice between two situations, one providing a constant delay to food and the other a set of variable delays of known duration. Then the constant delay is varied in order to discover the duration of the constant value that is equally preferred to the set of variable delays. Finally, one tries to determine what rule for reexpressing the delays comprising the variable set results in the observed balance point. Killeen (1968) used this strategy to determine the best power-function reexpression of a set of variable delay intervals.

2.4.2. Moving averages

Suppose the values of the independent variable change over time. Although earlier values of the independent variable probably have some effect on behavior, the most recently experienced values are likely to have the greatest impact. To describe the effect, then, we need some way of updating the average value of the independent variable that preferentially weights values based on their recency. Such calculations are called *moving averages* or *running means*. Killeen (1981) described the properties of several kinds of moving averages. One, the Exponentially Weighted Moving Average (EWMA), has wide applicability. For EWMA:

$$A_N = \beta X_N + (1-\beta)A_{N-1}$$

where A_N is the updated mean, A_{N-1} is the previous value of the mean, X_N is the new (i.e., most recent) value, and, β is a weighting factor, ranging between 0 and 1.0. The value of B determines how much the most recent event affects the updated mean. If B is close to zero, the new event will have only a small effect on the updated mean because the term to the left of the plus sign will be near zero. But if B is close to 1.0, the term on the right of the plus sign will be near zero, and so the new event will have a large effect on the updated mean. Intermediate values of B imply intermediate effects of the recent events. One can use EWMA as a model and determine the value of B that results in the best description of the behavioral adjustment to changing values of the independent variable (see Killeen, 1981)

3. **Evaluating a mathematical model**

To determine how well a given equation describes the data, one can plot the equation through the set of points and see whether it represents the trend well. If deviations appear random, those deviations may reflect the effect of uncontrolled variables. If those deviations are small, the function gains support as providing a good description. If, however, the deviations are systematic, even if small, then the model is not describing some systematic effect of the independent variable, and so the model is flawed in some way. (Judgments about the seriousness of the flaw will depend on a variety of considerations including the available alternatives and the conceptual underpinnings of the model (e.g., see Section 3.3).)

Visual and computational techniques are available to assess both the size and the systematic nature of deviations from some function (e.g., Mosteller and Tukey, 1977; Tukey, 1977; Kleinbaum et al., 1988) Some techniques evaluate mainly the size; others evaluate the extent to which the deviations are systematic.

In determining whether or not the deviations are systematic, it helps if the function is linear because deviations from a straight line are easy to detect. Thus, if the function being considered has a simple linear transformation, it is useful to reexpress the variables in the way implied (see Section 2.2 above, and Tukey, 1977)

3.1. Goodness-of-fit criteria (e.g., percent variance accounted for)

The square of the Pearson correlation coefficient, r^2, indicates the proportion of the total variance accounted for by the best-fitting straight line. Other curve-fitting techniques provide analogous measures. Clearly, a model that accounts for only a small proportion of the variance is not likely to be very effective. But researchers sometimes report values of r^2 as if they were paramount for judging the model's adequacy. In fact there are many reasons to view proportion of variance accounted for as only one of a number of criteria.

First, the proportion of variance measure does not indicate whether the deviations are systematic or unsystematic. Fig. 3 illustrates the difference between these two kinds of deviations. The data in the top row of Fig. 3 are the same as those in the bottom row of Fig. 2. But in Fig. 3 the best fitting straight lines are drawn and the r2 values indicated. Not surprisingly, the straight line accounts for a relatively small proportion of the variance in the linear plot (left-most panel). But the best fitting lines in both the middle and right-hand panels account for a fairly high proportion of the variance. Although the r2 value is higher in the right-hand plot, supporting a power function over an exponential function, the difference between those r2 values is too small to provide a firm basis for choosing one function over the other. But determining whether the deviations are systematic might help. One way to determine this is to plot the residuals, the difference between the obtained Y-value and the predicted Y-value derived from the best fitting function. Then these residuals are plotted over the X-axis values (or over the predicted Y-values) (e.g., Mosteller and Tukey, 1977; Tukey, 1977; Kleinbaum et al., 1988). If the deviations are unsystematic, then the residuals will scatter around a horizontal line. But if the deviations are systematic, then those residuals will vary in an orderly way. The residuals for the data in the top row are shown in the lower row of Fig. 3. These residual plots indicate that the deviations are indeed systematic for the linear and exponential functions (i.e., they are bitonic in form) but not for the power function. (Whether or not a power function is accepted as an adequate description would depend on additional considerations, as discussed in Section 3.3).

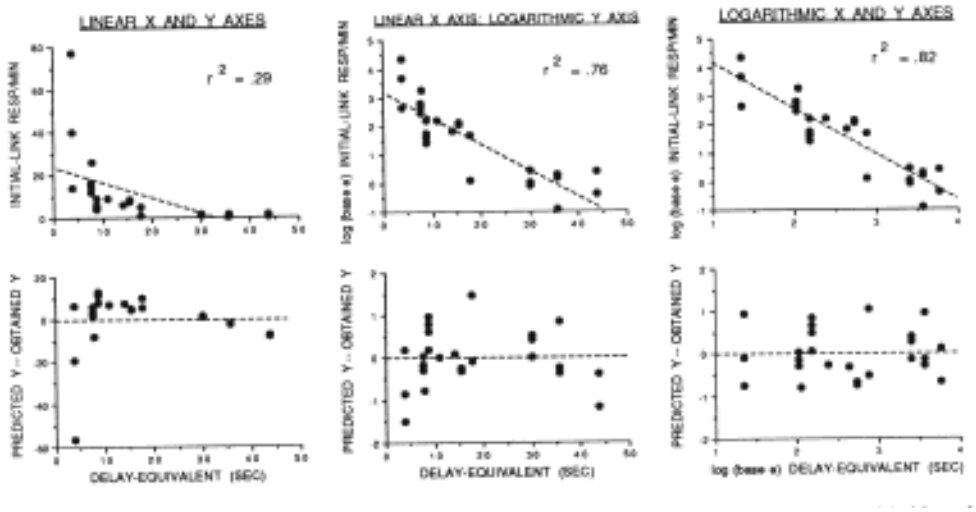

Fig. 3. *Top row:* Scatter plots from the bottom row of Fig. 2 with the best fitting straight lines added based on minimizing the squared deviations (Pearson product-moment). (Adapted from Shull et al., 1990, Pigeon 3819, with permission of the publisher.) *Bottom row:* The residuals plots.

A second potential problem with 'blindly' accepting r^2 values to support a model is illustrated in the left-hand panel of Fig. 4. In this plot, eight of the nine points appear to scatter around a decreasing function. The remaining point, an *outlier,* falls on a positive function relative to the other eight. Regression analyses are based on relative variances, and so a single extreme point can dramatically affect the outcome. Here the single outlier results in a positive best fitting function which accounts for 75% of the variance in the set of data. Common sense, however, suggests that the extreme data point may not be comparable with the others and that we ought not take the results of the regression analysis at face value. Textbooks on regression analysis (e.g., Kleinbaum et al., 1988) discuss statistical tests to aid judgments about whether or not a given point should be considered an outlier.

For similar reasons, it is important to consider whether or not the independent variable is evenly distributed over the X-axis. An effective strategy usually is to sample a wide range of values of the independent variable but to concentrate values in the region where the function of interest is changing the most. In that region the various alternative functions are likely to be most different (see Fig. 1), and so it is important to determine the shape in that region as thoroughly as possible. Yet, if the majority of values are bunched at one end with a few others distributed toward the other end of the range, then the effects can be like that produced by an outlier: the few extreme points can have a disproportionately large effect on the regression analysis. One needs, then, to be especially sensitive to the possible effects of spacing on parameter estimates. The X-axis spacing can be a particular problem when the values of the independent variable are reexpressed. Suppose the experimenter selected values of an independent variable that were evenly spaced along a linear X-axis (e.g., 10, 20, 30, 40, 50 and 60). But if these values are converted to logarithms for the purpose, say, of fitting some function, then the values of the reexpressed independent variable will be spaced progressively closer together, moving left to right along the X-axis. (See Wetherington and Lucas, 1980, and McDowell, 1981, for a discussion of related issues in connection with fitting Herrnstein's equation to a set of data.)

A third problem is that the range of variables along the X-axis can have a profound effect on r^2 values. The scatter of points in the right-hand panel of Fig. 4 appears to follow a linearly decreasing trend (in double logarithmic coordinates). The data are the same as those in the upper right-hand panel of Fig. 3 (Pigeon 3819 in Shull et al., 1990). In fact, the best fitting straight line accounts for 82% of the variance in the full data set, suggesting a moderately good fit. But suppose the experiment had sampled the independent variable only over the range of values bracketed by the vertical lines (i.e., the closed points). If so, only a very weak trend would have been apparent based on the Pearson technique ($r^2 = 0.17$), and the conclusion about the adequacy of the equation for describing the data could have been quite different. Again, the value of r^2 depends on relative variance. So if the independent variable covers only a small range of values, it may contribute little to the variance in the data relative to 'noise' even if it exerts a relatively potent effect when a larger range is considered.

Finally, if the data set contains only two points, the best fitting straight line will account for 100% of the variance. Thus, to have a meaningful test, the data set has to contain substantially more data points than there are free parameters to estimate. Unfortunately, there is no simple, absolute guide as to how many data points are sufficient.

An equation that precisely describes a lot of data and that contains few free parameters is likely to be judged superior. And, in fact, quantitative formulas have been developed to compare models based on those three factors: the average size of the deviations from predicted values, the number of data points, and the number of free parameters (e.g., Akaike, 1974, as described by Davison and McCarthy, 1988, pp. 113-114). Although such formulas sometimes can be helpful, relying too heavily on them can be dangerous. The fact is that evaluating a model requires good judgement based on careful consideration of many factors (e.g., Baum, 1983).

3.2. Accuracy versus generality

A different kind of issue about the importance of variance accounted for should be considered. Should one aim primarily for the equation that most accurately describes the pattern of results in the particular set of data at hand? Consider again the full set of data points in the right-hand panel of Fig. 4. Although an equation could be found that would connect all the points, thus accounting for 10076 of the variance in the data set, it would be a very complex one. In contrast, a linear equation provides a very simple description of the trend (in logarithmic coordinates). Yet the linear equation is inaccurate because no single straight line can connect all, or even most, of the points.

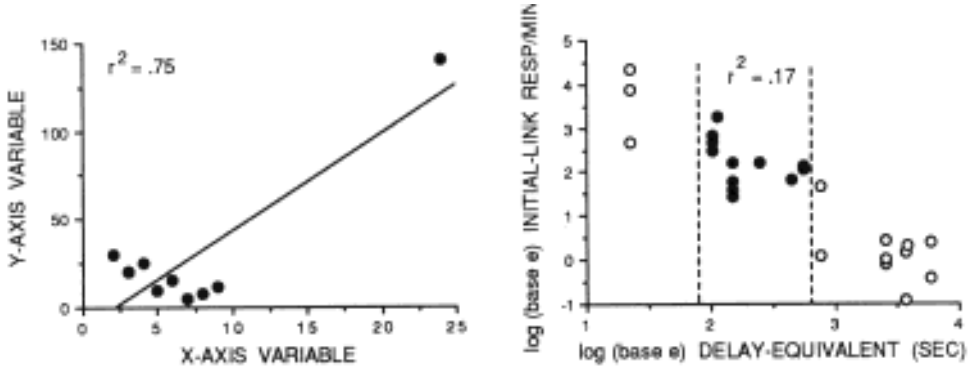

Fig. *4. Left panel:* Scatter plot (made-up data) showing the effect of an outlier. *Right-panel:* Scatter plot showing the effect of restricted sampling along the X-axis. (Adapted from Shull et al., 1990, Pigeon 3819, with permission of the publisher.)

Which equation should we choose? The answer depends on how we weight the answers to the following questions: Which equation better describes other sets of data obtained under similar and different conditions? Which equation has the fewest free parameters? And for which equation do the parameters relate most clearly to manipulable variables and basic theoretical concepts? An equation generated with the sole criterion of providing the best fit to a particular set of data is unlikely to provide a general, integrative description.

Herrnstein's hyperbolic equation came to dominate the field not because its fit to Catania and Reynolds' (1968) data was demonstrably superior. A power function probably would have fit about as well. But the hyperbolic function was derived from an equation developed for other data sets (i.e., matching in concurrent schedules), and so its parameters made sense (see Section 2. 1.2).

Estes (1959) discussed a related issue in connection with the development of stimulus sampling theory. He was concerned with describing data on the acquisition of barpressing in rats:

> Any number of simple mathematical functions can be fitted to the observed acquisition curves. ...we see the result of fitting a single empirical curve with two different mathematical functions. If we require only descriptive adequacy, then our choice of functions is clearly going to be quite arbitrary. If we are not willing to settle for an arbitrary choice, then some further reflection is in order.

> Why are we so reluctant to make an arbitrary choice of functions? Obviously because we are hoping eventually to achieve more than an accurate description of acquisition in one particular situation. We would like to arrive at a description which will hold for a variety of situations. And we would like to find that the parameters in our functions have theoretical significance, i.e., that their values reflect manipulations of variables in some reasonable way. These aspirations bring us up against a conservation law which has been recognized more or less explicitly since the time of John Locke to hold for all theoretical undertakings. Theories transform information; they do not create it. If we hope to get more out of our mathematical model than a description of a particular set of facts, then we will have to take a broader range of facts into account when formulating the model (pp. 389-390).

4. **Examples of applications of mathematical description**

4.1. Specification of the dependent variable

As Skinner (1935) noted, the search for the effective dimensions of responding (and of the environment) is an iterative process. The specifications are progressively modified until there is no further improvement in the orderliness of the relationship. Mathematical description can aid in this effort to specify the dependent variable better.

4.1.1. Response rate as composed of burst and pause modes

In operant conditioning experiments the experimenter usually measures responses as brief, discrete events that occur in time. Temporal distributions of responses sampled under different conditions or over different intervals of time under the same condition can differ from each other along many different dimensions. One such dimension is the average rate of a response (number of response/time available to make the response). A critical question is whether average rate is the dimension of responding that is related most simply to significant classes of environmental variables.

Skinner (1950, 1953, 1957) described operant behavior as varying in tendency, or in its probability of occurrence. Intuitively, response tendency should vary as a function of such variables as reinforcement and nonreinforcement, the amount and frequency of the reinforcer, properties of the discriminative stimulus, and motivational variables. To determine these controlling relationships precisely, we need to measure some property of responding that varies in a straightforward way with these classes of variables. Average response rate sometimes shows the appropriate orderliness (e.g. Herrnstein, 1970), but often it does not (cf. discussion by Nevin, 1979, especially pp. 148-150; Shimp, 1975). This failure is troublesome.

Perhaps the problem is that response rate is not a unitary dimension. Suppose that different aspects of the temporal distribution of responses are controlled by different classes of environmental events. If so, the overall average response rate would be a complexly determined composite. It may be possible, however, to dissect out the components of the temporal distribution, thereby revealing more elementary relationships.

Suppose, for example, that responding occurs in bursts, where the time between responses within a burst is controlled by one set of variables and the time between bursts is controlled by another set (e.g., Gilbert, 1958; Though, 1963; Premack, 1965; Williams, 1968; Pear and Rector, 1979; Nevin and Baum, 1980; Wearden, 1983). Assuming two different responding modes, the average rate of the response can be described as:

$$Total\ \mathrm{Re}\,spRate = \frac{Total\ \mathrm{Re}\,sp}{Totaltime}$$

$$\frac{\underline{(Resp\ rate\ in\ burst)\ x\ (Mean\ burst\ duration)\ x\ (Number\ of\ bursts)}}{Total\ time}$$

More formally:

$$B = \frac{P}{T} = \frac{FLN}{T} \tag{3}$$

where B refers to the total average rate of the response; P refers to the total number of responses; T refers to the total time that the response could occur; F refers to the average rate of the response during the bursts; L refers to the mean duration of the burst; and N refers to the number of times during the observation period that a burst was initiated. In words, the numerator in the right-hand expression says that the total number of responses is the product of the response rate during the burst, the average amount of time spent in the burst, and the number of times the burst was initiated.

We can consider, further, how to express the number of bursts. First, think of a cycle consisting of the time between bursts and the duration of the burst. The number of bursts *(N)* is the number of such cycles during the total session time (T). The mean time between the end of one burst and the start of the next is simply the reciprocal of the mean rate of burst initiations (i.e., 1/I, where I refers to the rate of burst initiations). (The rate of burst initiations would be the number of bursts divided by the total time minus the time spent in the bursts.) For example, if the rate is three initiations per min, the average time between initiations is 20 s. Thus, the cycle time is [(1/I) + L] And so the number of burst initiations is given by:

$$N = \frac{T}{\left(\frac{1}{I}\right) + L} \tag{4}$$

We can substitute the right-hand term in Equation 4 for N in Equation 3, which yields:

$$B = \frac{FL\left[\frac{T}{\left(\frac{1}{I}\right) + L}\right]}{T}$$

and which can further be reduced to:

$$B = \frac{FL}{\left(\frac{1}{I}\right) + L} \tag{5}$$

If the independent variable of interest changes only the rate of initiating bursts, without affecting the other components of the right-hand term, then the overall average response rate would change in the same direction. To appreciate this point, consider the denominator of Equation 5. As the rate of initiations (I) increases, 1/I decreases, and so B (the overall average response rate) will increase.

It may be worth emphasizing how the rate of initiating bursts should be calculated. A burst cannot be initiated during the time that it is occurring, only when it is not occurring. Thus, the rate of initiating bursts would be the number of bursts divided by the total time minus the time spent in the burst [i.e., *I=N/(T—(NL))]*.

With some further rearranging, Equation 5 becomes:

$$B = \frac{FI}{I + \left(\frac{1}{L}\right)} \tag{6}$$

If the response rate during the burst (F) and the mean duration of the burst *(L)* were constant, then Equation 6 would be a hyperbolic function, of the general form, $y = kx/(x + c)$, where x and y are variables and *k* and c are constants.

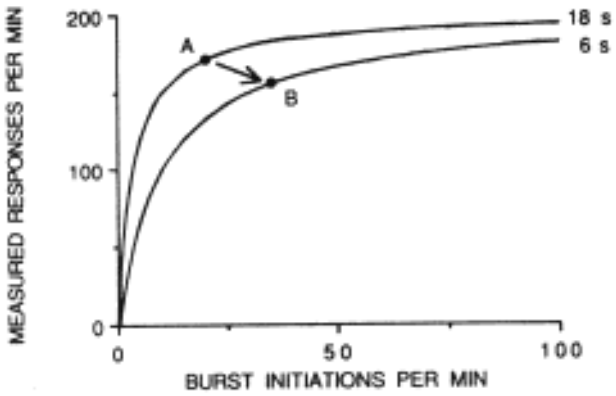

Fig. 5. The mean overall rate of the measured response (responses per min) plotted over the rate of burst initiations (initiations per min) based on Equation 6. The mean response rate during the burst was set at 200 per min. Each of the two curves assumes a different mean burst duration (6 or 18 s).

Fig. 5 plots the average overall response rate as a function of the rate of burst initiations according to Equation 6. Two functions are shown, corresponding to two different mean burst durations *(L* = 6 s and *L = 18 s).* Note that the functions are negatively accelerated so that the average overall response rate becomes progressively insensitive to changes in the initiation rate as initiation rate increases. Further, this insensitivity becomes more pronounced with progressively longer burst durations. An implication is that a variable that might have a strong and systematic effect on the rate of initiations will have little or no effect on the average overall response rate except under conditions where response rate is very low.

Suppose, further, that the independent variable affected other components of the expression as well as the rate of initiating bursts. If so, then the relation between overall average response rate and the rate of initiations would be even more complex. In fact the two rates could change in opposite directions if the variable increased the rate of burst initiations but decreased either the response rate within a burst (F) or the average burst duration *(L).* Such an effect is illustrated in Fig. 5 by the arrow showing a shift from Point A to Point B.

The discussion to this point has been to show how average overall response rate can be described formally as a composite of separate properties. An analysis of this sort is most likely to be useful if the separate properties are related to different classes of controlling variables. That possibility can be expressed more formally. We begin with the general statement that each of the components of response rate identified in Equation 6 is some function of environmental variables (E) which can include natural selection in the evolutionary past. And so,

$$F = f_1(E_1)$$
$$L = f_2(E_2)$$
$$I = f_3(E_3)$$

Substituting these into the equation for overall response rate (Equation 6) gives,

$$B = \frac{f_1(E_1) x f_3(E_3)}{f_3(E_3) + \left(\frac{1}{f_2(E_2)} \right)} \qquad (7)$$

Perhaps one or more of the components of overall response rate, such as burst- initiation rate, is simply and strongly related to classes of variables that we think should strongly affect response tendency in a fundamental way (e.g., rate or amount of reinforcement, deprivation, and variations of the discriminative stimulus). That is, $f_3(E_3)$ might be a simple one. But that simplicity will not show up in the overall measure of response rate if, in fact, overall response rate is composed of separately controlled components along the lines suggested here. The response rate within a burst, $f_1(E_1)$, might depend mainly on the differential reinforcement of response rate and on phylogenic variables.

The point of this discussion is not to advocate any particular representation of response rate but to illustrate an approach based on mathematical description. Such description can help one keep track of what the possibilities are, given certain assumptions about the possible composition of response rate. But they do not answer such difficult experimental questions as how to define and count 'bursts' (cf. Gilbert 1958; Pear and Rector, 1979). An experimental analysis is needed to establish the utility of the description.

4.1.2. Latency and interresponse-time distributions

What do we mean when we say that operant behavior is emitted and that variables influence the probability of emission? One interpretation can be illustrated with an admittedly crude analogy. Imagine a pot of boiling oatmeal on a stove with a device above the pot that records 'spatters' from the bubbles. Certain variables, such as the heat level under the pot, will determine the rate of spattering. But the exact moment of a spatter is determined by events (e.g., convergence of particular molecules) that are not related in any one-to-one manner with the molar variables like heat level.

In this analogy the spatters correspond to operant responses (emitted) and the heat level corresponds to the classes of variables that affect the emission rate of operant behavior such as rate of reinforcement, deprivation level and characteristics of the discriminative stimulus. The example is intended to illustrate a system that has a probabilistic, or stochastic, output. Manipulable variables (e.g., heat level or rate of reinforcement) determine the average emission rate but not the moments of occurrence. A powerful descriptive language derived from probability theory has been developed for prototypic stochastic systems (McGill, 1963; Luce, 1986) and should apply to operant behavior if operant behavior is related stochastically to classes of environmental variables.

Of particular interest is the time elapsed from some event until a response occurs, that is, response *latencies.* If the time is measured from the end of one response unit until the beginning of the next, the latencies are called *interresponse latencies* or *interresponse times.* For the following discussion, it does not matter whether the latency is timed from the previous response or from some other event (e.g., from reinforcement). What is important is that the latencies vary in characteristic ways that may reveal the nature of the controlling relationships (cf. McGill, 1963; Fagen and Young, 1978; Luce, 1986; Clifton, 1987).

Suppose that the probability of a response is constant regardless of elapsed time since the start of the latency (e.g., p=0.2 per bin, where bin refers to the time unit for recording latencies - i.e., the class interval). What should be the shape of the expected frequency distribution of latencies? Assume that we recorded the first 1000 latencies. The expected number of latencies that ended in the first bin would be the probability of a response during that bin (0.2) multiplied by the number of times the response could have occurred during that bin (1000): i.e., 0.2 (1000)=200. The expected number of latencies ended in the second bin would likewise be the probability of a response during that bin (0.2) times the number of opportunities to respond during the second bin. But the number of opportunities would not be 1000, since any latency that ended during the first bin would have eliminated an opportunity to respond during the second bin. Thus, the number of opportunities would be (1000— 200), or 800, and so the expected frequency in the second bin should be 0.2 (800)= 160. The expected frequency in the third bin would be calculated similarly: 0.2 (800 - 160)= 128. The theoretical calculations would proceed the same way if the constant probability per bin was 0.4 instead of 0.2 except that the values would, of course, be different.

The relative frequency distributions that would result from two different constant response probabilities (0.2 and 0.4) are shown in the upper left panel of Fig. 6. Relative frequencies are simply the expected number of latencies in a given bin divided by the total number of latencies. Even though the response probability is constant, the relative frequency function decreases continuously because there is a decreasing number of opportunities in each successive bin. Although the functions are similar in form, the function generated by the higher probability drops more sharply. Thus, both probability values produce a distribution of latencies, some long and some short. But a lower response probability generates relatively more long latencies.

The logic of these calculations can be generalized. There is an opportunity to respond in the ith bin only when the latency has not ended in an earlier bin. The probability that the latency has not ended in the first bin is (1-p), where p is the probability of a response during a bin and, for this example, is constant. Thus, the number of opportunities in the second bin is *N(1-p),* where *N is* the total number of latencies (N). Similarly, there is an opportunity to respond during the third bin only when the latency has not ended during either the first or the second bin. The chances of two nonoccurrences in a row is (1-p) (1-p), and so the number of opportunities to respond during the third bin is *N(1-p) (1-p).* More generally, the opportunities to respond during the ith bin, F_i, is:

$$F_i = N(1-p)^{i-1} \tag{8}$$

Then, the frequency in the ith bin, f_i, is:

$$f_i = N(1-p)^{i-1}\,p \tag{9}$$

which is a geometric decreasing progression; the frequency is a constant proportion of the opportunities remaining. (Relative frequencies and relative opportunities would be obtained by dividing through by *N.)* (Note that the opportunities value is represented by an upper case letter (F), while the frequency is represented by the same letter in lower case (f). This representation is commonly used to indicate that the two functions are tightly related. The opportunities function is a backward cumulative function of the frequency function.

We often are interested in determining how the probability of a response varies as a function of elapsed time during the latency, which can be determined from a logarithmic transformation of the opportunities function. Plots showing the logarithm of the opportunities as a function of elapsed time (not log-transformed) are called *log survivor functions* (cf. Luce, 1986).

When the response probability is constant, the opportunities function (Equation 8) is a geometric decreasing function, which is similar to a negative exponential (see Section 2.2). The logarithmic transformation of the opportunities function is

$$\log(F_i) = \log(\text{l—p})\,(\text{i—1}) + \log(\text{N}) \tag{10}$$

Because the logarithm of a decimal is negative in sign, Equation 10 says that the log survivors decrease linearly as a function of elapsed time (i). This linearity should make some intuitive sense. A constant probability of responding (p) implies that a constant proportion of the remaining opportunities will be lost in each bin. Equal proportions, or ratios, correspond to equal distances on a logarithmic scale.

The key point is that if one plots the log survivor function based on real data, the slope will indicate how the response probability varies as a function of elapsed time since the start of the latency. If, for example, the log survivor function decreases linearly with elapsed time, then the probability of a response is constant. The steeper the slope, the higher the response probability. If, instead, the log survivor function curves downward (i.e., decreases at a positively accelerated rate), the probability of a response is increasing over elapsed time. Conversely, if the log survivor function decreases at a negatively accelerated rate, the probability of a response is decreasing.

A different kind of plot, one showing response probability as a function of time, is known as the *hazard function* (e.g., Luce, 1986) or the interresponse time per opportunity function (Anger, 1956). The

hazard function shows response probability directly. In contrast, response probability is estimated from the slope of the log survivor function. But the log survivor function is useful because it can be plotted quickly.

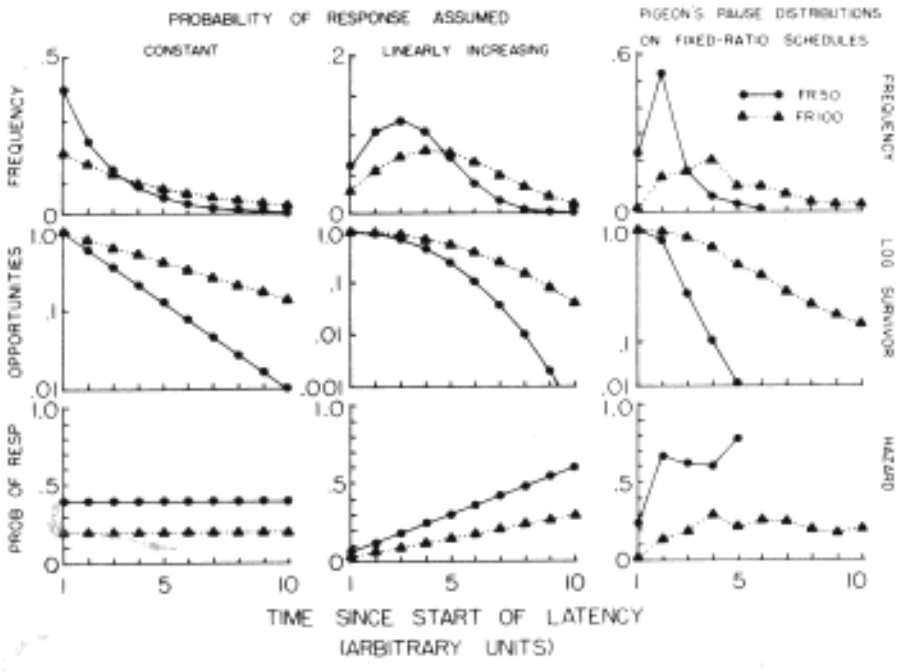

Fig. 6. Relative frequency distributions of response latencies (top row), log relative survivor functions (middle row), and hazard functions (bottom row) plotted over the class interval of elapsed time since the start of the latency. The graphs in the left and middle columns show data generated by assuming that response probability was constant (left) or increased linearly (middle) as a function of elapsed time. The graphs in the right column show response latencies (postreinforcement) from one pigeon studied under FR 50 and FR 100. (Adapted from Shull, 1979, with permission of the publisher.)

The left two columns of Fig. 6 show from top to bottom, theoretical frequency distributions, log survivor functions, and hazard functions based on different simple assumptions about response probability. In the far left column, the assumption was that response probability was constant over elapsed time (either 0.2 or 0.4). In the middle column the assumption was that the response probability increased linearly with elapsed time (two different linear functions). Note that the log survivor functions are linearly decreasing in the far left plot but are continuously curved downward in the middle plot. Also note that the frequency distributions in the middle column are bitonic instead of monotonic decreasing. The bitonicity is the product of a rising response probability and a declining number of opportunities.

For these illustrations, we worked backward compared to what is done in empirical research. Here, we asserted values for the response probability and then calculated the expected frequencies and opportunities. In actual research, we obtain the frequencies of latencies and from them estimate the response probabilities by dividing the frequency in each class interval by the opportunities.

We may illustrate these calculations by considering the response latencies generated by a pigeon under fixed-ratio (FR) schedules of food reinforcement. The FR value was 50 for a block of sessions and then 100 for another block. The latencies were timed from the completion of one FR 'run' until the start of the next. During each of the last 5 sessions of an FR value, the latencies were recorded in sequence on paper tape from which the frequency and opportunity distributions were constructed (5-s bins). The number of opportunities in a particular bin is the number of latencies as long or longer than the lower bound of the bin. Finally, the frequency per opportunity values were calculated by dividing the number of latencies ended in a bin by the number of opportunities to respond in that bin. These functions are shown in the right-hand panels of Fig. 6.

The log survivor functions appear roughly linear over much of their range after an initial period of curving downward. They indicate that the response probability is fairly constant after an initial rise. Furthermore, because the log survivor function is steeper under FR 50 than under FR 100, the response

probability is higher under the smaller (more favorable) FR schedule. These variations in response probability can be seen more directly in the probability functions (the hazard functions).

The appearance of the log survivor functions for FR schedules is similar to that found with response-initiated fixed-interval (FI) schedules (e.g., chain FR I FI schedules) but is different from those found with FI schedules where the interval is timed from the last reinforcer delivery. Under FI schedules, the log survivor functions are continuously curved downward. These differences, which are not apparent in the frequency distributions of latencies, may suggest different temporal control relations under FR (and response-initiated FI) schedules versus under FI schedules (cf. Shull, 1979; Capehart et al., 1980).

With these sorts of plots, one can see the extent to which responding approximates a simple random emitter. As discussed earlier in this section, a simple random emitter implies a constant response probability (i.e., a horizontal hazard function and a linearly decreasing log survivor function). It is clear from the data in the right-hand column of Fig. 6, for example, that the response probability is not constant during the latency under FR schedules, although it may approach constancy after an initial rise.

Deviations from a simple random emitter can occur for a number of reasons (cf. McGill, 1963; Luce, 1986). For example, the measured response might be the end product of a chain of activities (e.g., Killeen and Fetterman, 1988). Each of the activities comprising the chain might have a constant probability of terminating. But if so, the probability of completing the whole chain will rise and then level off. Alternatively, the emission probability might be under discriminative control of elapsed time since the start of the latency. Or the temporal distribution of responses might be a composite of two or more different emission processes, such as discussed in Section 4.1.1. In such cases, there might be a sharp bend in the log survivor functions (cf. Clifton, 1987). For some prototypic cases of these types of processes the mathematical description has been worked out (cf. McGill, 1963; Fagen and Young, 1978; Luce, 1986; Clifton, 1987). It may be possible, therefore, to make inferences about the structure of responding and, perhaps, about the kinds of controlling relationships based on the match between actual latency distributions and the theoretical functions of interest. The treatment of these issues is, it must be noted, quite complex (cf. Luce, 1986 for a thorough discussion).

One additional point requires comment. In the discussion to this point, elapsed time was treated as divided into discrete class intervals. A parallel mathematical description is available for the limiting case where the class interval size is conceptualized as vanishingly small so that elapsed time can be represented as a continuous variable. There are analytical advantages to developing models based on continuous functions (e.g. the availability of the techniques of calculus), and one is likely to see descriptions expressed in that form (e.g., McGill, 1963; Killeen, 1975; Luce, 1986). For example, the analog of the geometric decreasing function would be:

$$f(t) = \lambda e^{-\lambda t}$$

where λ is a rate constant analogous to p in the geometric function; *t is* elapsed time analogous to i in the geometric function; and *e is* the base of the natural logarithms. In log form, $\ln(f(t)) = -\lambda t + \ln(\lambda)$ which is linear. (Recall that the e-term drops out because the log of the base is 1.) More complex functions describing probabilistic events in time usually have their discrete- interval counterparts that are harder to work with mathematically but that are often easier to appreciate conceptually. With some diligence, it is often possible, even for the mathematically unskilled, to construct some approximation to the discrete-interval analog.

4.2. Specification of the independent variable

A researcher manipulates concrete environmental events. The particular manipulations differ from each other along many different dimensions. Some of those dimensions may influence behavior whereas others may not. The task is to discover the dimensions that systematically affect behavior. More formally, we are seeking an effective specification of *E* (an environmental variable) in the general formula, *B=f(E).* A few examples are described briefly.

4.2.1. The temporal distribution of reinforcers

A pigeon's choice between two stimulus situations depends on the temporal distribution of food reinforcers obtained in the two stimuli. But what exactly about the temporal distribution of reinforcers controls choice? One possible dimension is the amount of food reinforcement per time spent in the situation - that is, the arithmetic rate of food reinforcement, or *MA=R/T*, where *MA* refers to the arithmetic rate of reinforcement; *R* refers to the number of reinforcers obtained in a situation; and T refers to the time spent in the situation. But choice is not related consistently to *(R/T)* when the temporal distributions of reinforcement are varied systematically (e.g., McDiarmid and Rilling, 1965; Killeen, 1968; Mazur, 1986; Shull and Spear, 1987).

In one study (Shull and Spear, 1987) pigeons could remain in one stimulus (Stimulus A) where they received food on a VI schedule or they could switch to another stimulus (Stimulus B). Over conditions, several aspects of the temporal distribution of reinforcers during Stimulus B were varied: the duration of this second situation, the number of food reinforcers delivered, and the delays to those food deliveries timed from the switch. The measure of responding was the rate of switching into the second situation. The left-hand panel of Fig. 7 shows, for one of the pigeons, the switching rate (logarithmic scale) plotted over the arithmetic rate of reinforcement in Stimulus B. The plot of points in that panel shows no apparent order. One might conclude either that the independent variable has no systematic effect or that uncontrolled variables mask any systematic effect. But the data in the right-hand panel of Fig. 7 suggest a different conclusion. Switching rate is plotted as a function of a different property of the temporal distribution of reinforcers, namely the sum of the immediacies, $\Sigma\,(1/D_i)$, where D_i indicates the delay to the ith (i.e., the first, the second, the third etc.) food reinforcer following the switch, where all delays are timed from the switch (McDiarmid and Rilling, 1965; Mazur, 1986; Shull et al., 1990). Switching rate appears to vary as an orderly function of the sum of the immediacies.

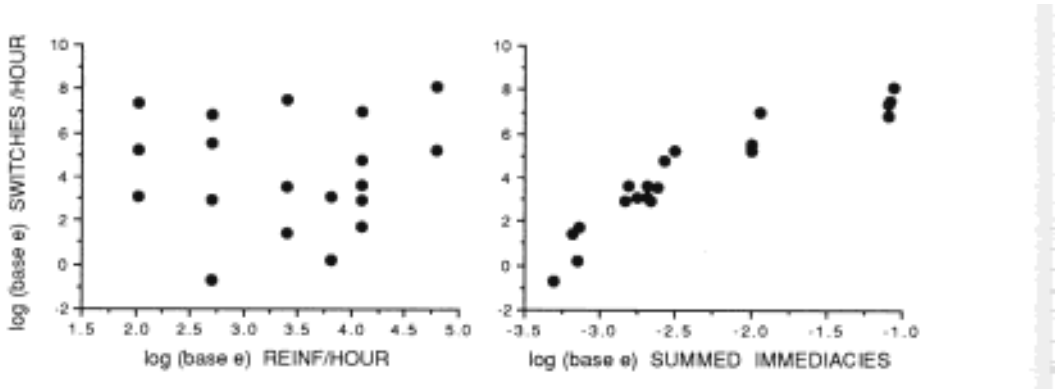

Fig. 7. Rate of switching to Stimulus B as a function of two ways of specifying the temporal distribution of food reinforcers in Stimulus B. (Adapted from Shull and Spear, 1987, Pigeon 3819, with permission of the publisher.)

This example illustrates several points of relevance to a general consideration of mathematical description. First, behavior can appear unrelated to a set of concrete manipulations when those manipulations are described one way but can appear highly related to those same manipulations when those manipulations are described in a different way. Second, by specifying the dimensions (i.e., rate of reinforcement or sum of the immediacies) in quantitative terms, it was possible to order the experimental conditions along the X-axis in different ways and thus see which dimension had the most systematic effect on behavior. Third, one may be able to determine rather precisely what dimensions of the environment influence behavior without precisely specifying the function between that dimension and behavior. It may be sufficient to say merely that the dependent variable is an ordinal function of the independent variable dimension.

Other examples of using mathematical quantification to determine how best to specify the independent variable include: Fantino and Abarca's (1985) analyses of the temporal variables that influence choice in laboratory analogs of foraging; Killeen's (1968) work on finding the transformation rules for determining the conditioned reinforcement values produced by VI versus FI schedules of food reinforcement; Killeen's (1981) analysis of how best to describe the dimension of recency (see Section 2.4.2); and, efforts to determine how the relational property of the environment known as the *contingency* should be specified (cf. Gibbon et al., 1974; Gibbon and Balsam, 1981).

4.3. Reciprocal determinism (feedback functions)

In operant conditioning the variable that is assumed to influence responding inevitably depends on responding. The rate of the reinforcer, for example, is assumed to influence the rate of the response. But because the reinforcer is contingent on the response, the reinforcement rate depends to some degree on the response rate. Thus, the dependent variable affects the independent variable. Such reciprocal determinism may appear to prohibit an identification of simple controlling relations between behavior and the environment. That appearance is false, however (cf. Zuriff, 1985, pp. 108-110). There are mathematical techniques that can help disentangle the reciprocal controlling relationships.

Two functions are required. One, *the feedback function,* describes how the controlling aspect of the environment is altered by behavior. For example, a feedback function might describe how the reinforcement rate will depend upon the response rate. It is important to appreciate that the feedback function does not predict behavior. It is strictly a description of some aspect of the environment - a complex relational aspect, or a contingency (cf. Weingarten and Mechner, 1966). It describes the way the *environment* changes under certain conditions. The second function is the *behavior function,* which describes how some property of behavior is affected by the environmental variable. For example, it might describe how the rate of reinforcement is assumed to affect the rate of responding. That function cannot be determined empirically in the total absence of a dependency between rate of reinforcement and the rate of behavior. Nonetheless, conditions can be arranged that progressively weaken the feedback constraint. For example, response rate affects reinforcement rate much less under VI schedules than under variable-ratio (VR) schedules. Thus, VI schedules are commonly used in preparations intended to determine the behavior function for the effect of rate of reinforcement on the rate of behavior. Confidence in the behavior function depends on a variety of considerations including how well it fits the data as conditions more closely approximate the ideal (feedback free) situation, the validity of predictions about behavior in other situations, connections with other theoretical developments, and so forth. Once the feedback and behavior functions are specified (perhaps only as hypotheses to be tested), they can be combined to derive equilibrium solutions. A simple illustration of this general approach follows (see also Baum, 1973, 1981; Nevin and Baum, 1980; Rachlin et al., 1981; McDowell and Wixted, 1988).

Under VR schedules, reinforcement is delivered when a specified number of responses has occurred. Thus, the faster the animal responds, the more quickly reinforcement occurs. Given a sufficiently long averaging interval, the reinforcement rate on VR schedules is proportional to the response rate, with the reinforcement-to-response ratio *(I / VR)* determining the proportionality. That is:

$$R = \left(\frac{1}{C}\right) B(60)$$

where *R is* the reinforcement rate, B is the response rate, and C is the mean of the VR schedule (i.e., the mean response count per reinforcer). The 60 is simply a scaling factor due to the convention of expressing response rates in responses per min and reinforcement rates in reinforcers per h. Equation 11 is a feedback function. We may take Herrnstein's hyperbolic equation as an example of a behavior function (cf. McDowell and Wixted, 1988, for one of several alternatives). As presented earlier (Equation 2):

$$B = \frac{kR}{R + R_o}$$

Because the feedback function and the behavior function have the same two terms (B and *R),* they can be plotted in the same coordinates. Fig. 8 shows the feedback functions resulting from three different VR schedules. These are the three straight lines rising from the origin at the lower left. Also shown are two behavior functions based on the hyperbolic function with k= 100. The upper one is based on setting *Ro* equal to 10; the lower one is based on setting *Ro* equal to 50. The point where the feedback function intersects the behavior function is the *equilibrium point.* Thus, the response rate can be predicted, given the behavior function, by drawing a line horizontally from the intersection to the Y-axis. To appreciate

why this is so, consider what would happen, say, for the VR 60 schedule if response rate were below the intersection point. That response rate generates a reinforcement rate (via the feedback function) that maintains a response rate specified by the behavior function. But because the behavior function is above the feedback function at that point, the response rate should tend to increase. When the response rate increases, the reinforcement rate will increase also, causing a shift to the right on the behavior function. Eventually, the response rate will generate a reinforcement rate that will generate exactly that same response rate again. That point is where the functions intersect. Response rates above the equilibrium point will adjust in a similar fashion except in the reverse direction.

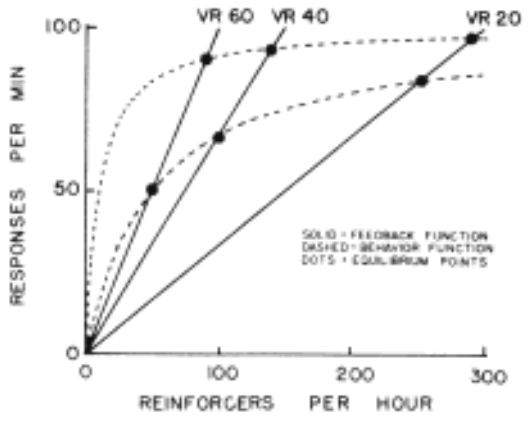

Fig. 8. Equilibrium solution for response rates on VR schedules.

The equilibrium points resulting from different VR sizes or from different behavior functions can be determined graphically, as just suggested (cf. McDowell and Wixted, 1988). It also may be possible to develop an equation for finding the equilibrium point. The goal is to generate an equation showing how response rate at equilibrium varies with, say, VR size. The key is to notice that at equilibrium the coordinate values of the two functions are equal. Thus, one writes both equations (the behavior equation and the feedback equation) in terms of reinforcement rate. Then one sets the two equations equal and solves for response rate. Visually, in terms of Fig. 8, you are moving along the X-axis until you reach the point where both functions have the same reinforcement rate - i.e., the point where they intersect. Then you go up to the intersection point and move left horizontally to get the response rate.

The feedback function (Equation l l) already is written in terms of reinforcement rate. The hyperbolic behavior function needs to be rewritten. With some rearranging of terms Equation 2 becomes:

$$R = \frac{BR_o}{k - B} \tag{12}$$

Now setting Equation l l to Equation 12 gives:

$$\left(\frac{1}{C}\right)B(60) = \frac{BR_o}{k - B}$$

and solving for *B*, the response rate, produces

$$B = k - CR_o\left(\frac{1}{60}\right) \tag{13}$$

which is the equilibrium solution. By this solution, response rate should be a linearly decreasing function of the VR size. That implication can be seen visually in Fig. 8 since the VR schedules represented by the feedback functions comprise a linear progression. Read the Y-axis values corresponding to the

intersections along a particular behavior function. Those Y-axis values will be spaced evenly, indicating a linear progression of response rates.

The point of this exercise is to show how, in principle, mathematical description can aid the analysis of controlling relations where the controlling variable (here reinforcement rate) depends, in turn, on the controlled variable (here the response rate). The discovery of feedback functions is straightforward conceptually but often difficult in practice. One needs to decide what environmental dimensions to represent in the equation. If, for example, one assumed that the average rate of reinforcement influenced responding, one would develop a feedback function that showed how rate of reinforcement depended on behavior under a given schedule. But if one assumed that some other dimension of reinforcement was important, then one would develop a feedback function that showed how that other dimension depended on behavior. For example, suppose that the temporal patterning of reinforcement, not just the average rate of reinforcement, was assumed to matter. Then the feedback function would need to describe how behavior affects the temporal patterning of reinforcers (cf. Nevin and Baum, 1980, for a related discussion). Regardless of the specific solution,‘the approach is to determine the equilibrium points based on the particular set of equations and then to see how well the data conform to those predictions.

4.4. Experimental tests based on nonobvious implications of mathematical models

Sometimes alternative models generate functions that are similar in shape despite being derived from very different conceptions of the process (see Fig. 1). It may be hard to decide among the models, then, unless one can find some set of circumstances where the different models imply clearly different outcomes. Mathematical description sometimes aids the discovery of such circumstances.

4.4.1. Example: indifference point analysis

Suppose, for example, that we are interested in describing the effect of reinforcement delay on responding. There is ample precedent for regarding the function as decreasing, negatively accelerated. Among the functions that have been proposed are the negative exponential and the reciprocal (or hyperbolic). Although they appear similar, they have different implications for understanding preference shifts between small immediate and large delayed reinforcers depending on the time of the choice (Ainslie, 1975, 1989). How should we test the adequacy of these two models? One approach would be to generate a set of data showing how response rate declined as a function of delay and then try to fit the two (or more) functions to that data set in order to see which one fits best. But the data might be too variable to discriminate between such similarly shaped functions.

An alternative strategy is to see if there are implications of the functions that can be exploited in an experimental test. Mazur’s work (e.g., 1987) is exemplary. Imagine giving a pigeon the following choice: one response produces a certain amount of the reinforcer (A_1) after a particular delay (D_1). The other response produces a larger reinforcer amount (A_2) delivered after a different delay (D_2). If the delays are equal (i.e., if $D_1 = D_2$), then choice will favor the second alternative, which provides the larger reinforcer. But if D_2 were progressively lengthened, at some point the negative effects of the longer delay will balance the additional positive effect of the larger amount. That point is the *indifference point.*

The predicted duration of D_2 at indifference depends on the function relating reinforcer effectiveness (V) to delay and amount. For each such function, one can show how the predicted duration of D_2 at indifference varies as a function of the duration of Di and the reinforcer amounts. These indifference functions are different enough for the different models to permit a clear experimental test. Table 1 shows the derivation for just two of the many possible function types. For the exponential function, the predicted duration of D_2 at indifference is a linear function of D_1, with slope of 1 and an intercept whose value depends on the magnitudes of the two reinforcers. For the reciprocal function, the predicted duration of D_2 at indifference is a linear function of D_1, with zero intercept and slope varying as a function of the ratio of the reinforcer amounts.

The experimental task, then, would be to find the indifference points under different durations of a standard delay and under different reinforcer amounts for the alternatives. The quality of the data certainly should be sufficient to determine whether changing the reinforcer amounts alter the intercept or the slope of the indifference function (cf. Mazur, 1987).

TABLE 1

Derivation of indifference-point functions based on two different possible functions relating the effectiveness of a reinforcer to its delay. V= reinforcer effectiveness; A = reinforcer amount; *D* = reinforcer delay; b is a curve-fitting constant; the subscripts identify the choice alternative.

Negative exponential	Reciprocal
Reinforcer effectiveness function	
$V_1 = A_1 e^{-bD_1}$; $V_2 = A_2 e^{-bD_2}$	$V_1 = \frac{A_1}{bD_1}$; $V_2 = \frac{A_2}{bD_2}$
At indifference, V' = V2, and s	
$A_1 e^{-bD_2} = A_2 e^{-bD_2}$, and $\ln(A_1) - bD_1 = \ln(A_2) - bD_2$	$\frac{A_1}{bD_1} = \frac{A_2}{bD_2}$
Solving for D_2 as a function of D_1 by algebraic rearranging gives the indifference-point function:	
$D_2 = D_1 + \frac{\ln(A_2) - \ln(A_1)}{b}$	$D_2 = \frac{A_2}{A_1} D_1$

This example illustrates how implications of mathematical descriptions can suggest experimental procedures that magnify differences between models. The structure of the mathematical language permits those implications to be generated readily and unambiguously (cf. Gibbon and Church, 1981; Vaughan, 1981, for other examples).

5. Conclusion

It may appear that mathematical models explain phenomena in some special sense. But such a view is unnecessary. One view of science, often attributed to Mach, is that what science (or at least the best science) does is provide efficient descriptions of functional relations (cf. Skinner, 1957, 1969; Marr, 1985, 1986; Smith, 1986). From this view, there is no 'deeper' kind of explanation. Descriptions expressed in a mathematical language may be especially precise, general, and efficient, and so may be especially useful. But they are not fundamentally different from common language descriptions. The development of successful mathematical description, like the development of any kind of effective description, requires such things as noticing possible similarities among phenomena, careful experimental analyses and clear thinking.

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