1.2 Gaussian, Row Reduction and Echelon Forms

$$\begin{aligned} x+2y+z &= 3 \\ 3x+6y-z &= 4 \\ 5x+10y+z &= 10 \end{aligned} \quad \leftrightarrow \quad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 6 & -1 & 4 \\ 5 & 10 & 1 & 10 \end{bmatrix}$$

Elementary Row Operations

replacement interchange scaling

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Elementary Row Operations

replacement interchange scaling

Strict Gaussian Elimination → Row Echelon Form

1. When possible, use top left spot in row 1 to eliminate the other column 1 terms via replacement
2. Ignore row 1 and use first nonzero spot in row 2 to eliminate the terms below it via replacement
3. . . . triangular (interchange as needed)

Leading Entries, Pivots and Pivot Columns

$$\begin{aligned} x + 2y + z &= 3 \\ 3x + 6y - z &= 4 \\ 5x + 10y + z &= 10 \end{aligned} \qquad \begin{bmatrix} \boxed{1} & 2 & 1 & 3 \\ 3 & 6 & -1 & 4 \\ 5 & 10 & 1 & 10 \end{bmatrix} \xrightarrow[r_3'=-5r_1+r_3]{r_2'=-3r_1+r_2}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ -3\cdot 1 + 3 = 0 & -3\cdot 2 + 6 = 0 & -3\cdot 1 - 1 = -4 & -3\cdot 3 + 4 = -5 \\ -5 & 10 & 1 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -4 & -5 \\ -5\cdot 1 - 5 = 0 & -5\cdot 2 + 10 = 0 & -5\cdot 1 + 1 = -4 & -5\cdot 3 + 10 = -5 \end{bmatrix}$$

We used the 1, a *leading nonzero entry* across row 1 to create 0s below it. It's spot is a *pivot position*. A *pivot column* is a corresponding column in original matrix. 1 is a *pivot* for x.

Leading Entries, Pivots and Pivot Columns

$$\begin{aligned} x + 2y + z &= 3 \\ 3x + 6y - z &= 4 \\ 5x + 10y + z &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 6 & -1 & 4 \\ 5 & 10 & 1 & 10 \end{bmatrix} \xrightarrow[r_3' = -5r_1 + r_3]{r_2' = -3r_1 + r_2}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ -3\cdot 1 + 3 = 0 & -3\cdot 2 + 6 = 0 & -3\cdot 1 - 1 = -4 & -3\cdot 3 + 4 = -5 \\ -5 & 10 & 1 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -4 & -5 \\ -5\cdot 1 - 5 = 0 & -5\cdot 2 + 10 = 0 & -5\cdot 1 + 1 = -4 & -5\cdot 3 + 10 = -5 \end{bmatrix}$$

We used the 1, a *leading nonzero entry* across row 1 to create 0s below it. It's spot is a *pivot position*. A *pivot column* is a corresponding column in original matrix. 1 is a *pivot* for x.

$$\begin{bmatrix} \textcircled{1} & 2 & 1 & 3 \\ 0 & 0 & \textcircled{-4} & -5 \\ 0 & 0 & -4 & -5 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 6 & -1 & 4 \\ 5 & 10 & 1 & 10 \end{bmatrix}$$

$$\begin{aligned} x + 2y + z &= 3 \\ 3x + 6y - z &= 4 \\ 5x + 10y + z &= 10 \end{aligned} \quad \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 6 & -1 & 4 \\ 5 & 10 & 1 & 10 \end{bmatrix} \xrightarrow[r_3' = -5r_1 + r_3]{r_2' = -3r_1 + r_2}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & -4 & -5 \end{bmatrix} \xrightarrow{r_3' = -r_2 + r_3}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -4 & -5 \\ -1 \cdot 0 + 0 = 0 & -1 \cdot 0 + 0 = 0 & -1 \cdot -4 - 4 = 0 & -1 \cdot -5 - 5 = 0 \end{bmatrix}$$

$$\begin{bmatrix} \textcircled{1} & 2 & 1 & 3 \\ 0 & 0 & \textcircled{-4} & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ row echelon form, pivots circled, 0s below

Solutions from Row Echelon Form after Gaussian

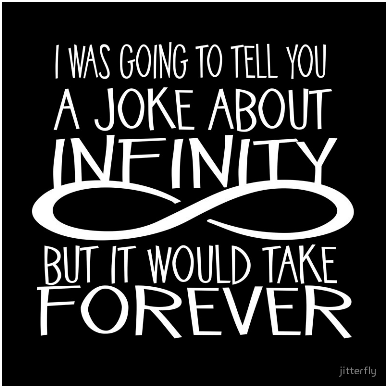
$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 6 & -1 & 4 \\ 5 & 10 & 1 & 10 \end{bmatrix} \longrightarrow \begin{array}{cccc} x & y & z & = \\ \end{array} \begin{bmatrix} \textcircled{1} & 2 & 1 & 3 \\ 0 & 0 & \textcircled{-4} & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- once in row echelon form, check for consistency by making sure we don't have any row with $0x + 0y + 0z = nonzero$
- if consistent, then identify pivots & free parameters, if any: no pivot in 2nd column, so y is a *free parameter*: $\boxed{y = t}$.
- back substitute variables attached to pivots, working from bottom up.
    - row 3: [0 0 0 0] represents $0x + 0y + 0z = 0$ no info
    - row 2: [0 0 -4 -5] represents $0x + 0y - 4z = -5$, so $\boxed{z = \frac{5}{4}}$
    - row 1: [1 2 1 3] represents $x + 2y + z = 3$, so

      $x = 3 - 2y - z = 3 - 2\boxed{t} - \boxed{\frac{5}{4}} = \frac{7}{4} - 2t$

Free Parameter Algebra and Geometry

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{7}{4} - 2t \\ t \\ \frac{5}{4} \end{pmatrix}$$



Free Parameter Algebra and Geometry

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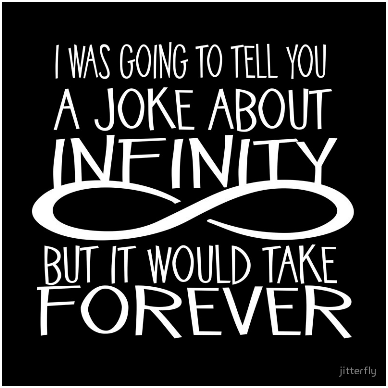


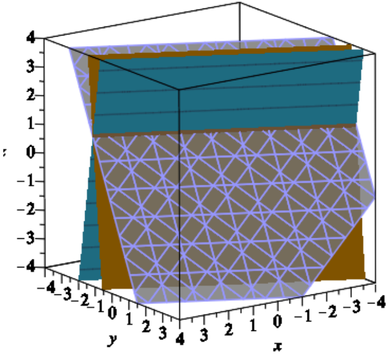


Image Credit: jitterfly

Row Echelon Form

A matrix is in *row echelon form* (ref) if

1. all nonzero rows are above any rows of all zeros
2. each leading entry of a row is in a column to the right of the leading entry of the row above it
3. all entries in a column below a leading entry are zeros

Maple: GaussianElimination(A)

ref solutions

Which augmented matrix is not in ref/Gaussian form?

a) $$\begin{bmatrix} 3 & 2 & 5 & 6 & 11 \\ 0 & 0 & 2 & -5 & 7 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

c) $$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) $$\begin{bmatrix} 3 & 2 & 5 & 6 & 11 \\ 0 & 0 & 2 & 0 & 7 \\ 0 & 0 & 4 & 0 & 14 \end{bmatrix}$$

ref solutions

Which augmented matrix is not in ref/Gaussian form?

a) $\begin{bmatrix} 3 & 2 & 5 & 6 & 11 \\ 0 & 0 & 2 & -5 & 7 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 2 & 5 & 6 & 11 \\ 0 & 0 & 2 & 0 & 7 \\ 0 & 0 & 4 & 0 & 14 \end{bmatrix}$

consider how many solutions, if any, and consider how many free variables, if any

Pivots, Free Variables, Solutions (if Any)

a)

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \\ \end{array}$$
$$\left[\begin{array}{ccccc} \boxed{3} & 2 & 5 & 6 & 11 \\ 0 & 0 & \boxed{2} & -5 & 7 \\ 0 & 0 & 0 & 0 & \boxed{2} \end{array}\right]$$

no solution, no concurrent intersections in $\mathbb{R}^4$

b)

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \\ \end{array}$$
$$\left[\begin{array}{ccccc} \boxed{3} & 2 & 5 & 6 & 11 \\ 0 & 0 & \boxed{2} & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

infinitely many solutions, a plane in $\mathbb{R}^4$ (x_2, x_4 free)

c)

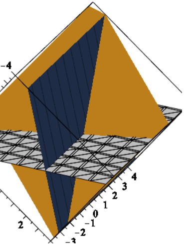
$$\begin{array}{cccc} x & y & z & = \\ \end{array}$$
$$\left[\begin{array}{cccc} \boxed{1} & 2 & -2 & 3 \\ 0 & \boxed{4} & 8 & 4 \\ 0 & 0 & \boxed{2} & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

unique solution, a point in $\mathbb{R}^3$

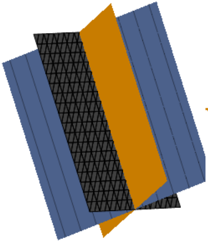
underdetermined, less equations than unknowns a) and b)
overdetermined, more equations than unknowns c)

Solutions and Pivots

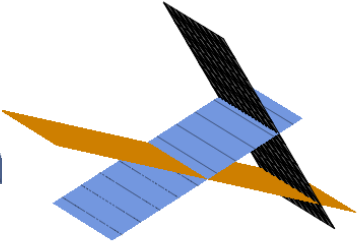
3 equations and 3 unknowns. How many pivot positions and pivot columns does each augmented matrix have?



1 solution
corner of room



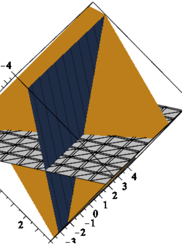
infinite solutions
book spine

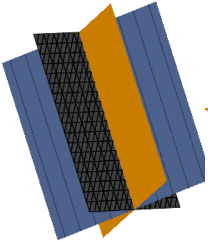


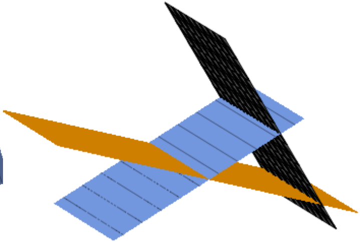
0 solutions
hands + table

Solutions and Pivots

3 equations and 3 unknowns. How many pivot positions and pivot columns does each augmented matrix have?







1 solution	infinite solutions	0 solutions
corner of room	book spine	hands + table
3 pivots	2 pivots	at least 1 pivot in =

any linear system has 0, 1, ∞ solutions

zeros followed by a non-zero in the equal column $\Longrightarrow$ inconsistent with no solutions. Otherwise, we are consistent and if every variable has a pivot then we have a unique solution while if at least one variable is missing a pivot in a consistent system then at least 1 free variable and infinite solutions.

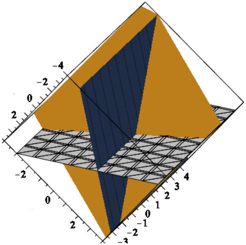
Solutions and Pivots

Can an underdetermined system with less equations than unknowns ever have a unique solution?

Solutions and Pivots

Can an underdetermined system with less equations than unknowns ever have a unique solution?

- No, as there must be a missing pivot as we have too few rows for there to be a pivot in each of the columns of the coefficient matrix resulting in no solutions or infinite solutions depending on consistency.
- We can have a unique solution with the same number of equations as unknowns or more equations than unknowns.



1 solution
corner of room

Reduced Row Echelon Form

A matrix is in *reduced row echelon form* (rref) if

1. all nonzero rows are above any rows of all zeros
2. each leading entry of a row is in a column to the right of the leading entry of the row above it
3. all entries in a column below a leading entry are zeros.
4. **the leading entry in each nonzero row is a 1**
5. **each leading 1 is the only nonzero entry in its column.**

Gauss-Jordan elimination
Maple: ReducedRowEchelonForm(A)

Moving from Gaussian to Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moving from Gaussian to Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_2' = \frac{1}{4} r_2} \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moving from Gaussian to Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & \boxed{4} & 8 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_2' = \frac{1}{4} r_2} \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{r_1' = -2r_2 + r_1} \begin{bmatrix} 1 & 0 & -6 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moving from Gaussian to Reduced Row Echelon Form

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 4 & 8 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_3' = \frac{1}{2} r_3} \begin{bmatrix} 1 & 0 & -6 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\substack{r_2' = -2r_3 + r_2 \\ r_1' = 6r_3 + r_1}} \begin{array}{cccc} x & y & z & = \end{array} \begin{bmatrix} \textcircled{1} & 0 & 0 & 13 \\ 0 & \textcircled{1} & 0 & -3 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

no inconsistencies [0 0 0 nonzero] and full pivots so unique solution $(x, y, z) = (13, -3, 2)$

Flops

A flop, for floating point operation, is a measure of $+, -, \times, \div$

- For an $n \times (n+1)$ matrix, it can take as many as $\dfrac{2n^3}{3} + \dfrac{n^2}{2} - \dfrac{7n}{6}$ flops to apply Gaussian elimination to reach the row echelon form (the *forward phase*).
- The *backwards phase* from row echelon form to reduced row echelon form (making use of all the zeros) can take another n^2 flops.

3×4 from ref to rref?

Describe the Algebra and Geometry of the Solutions

Describe the solution set to the *homogeneous* linear system with GaussianElimination(A)

$$\left[\begin{array}{c|c} & 0 \\ A & 0 \\ & 0 \end{array}\right] \xrightarrow{\textit{strict Gaussian}} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \\ \left[\begin{array}{ccccc} \boxed{3} & 0 & -3 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Describe the Algebra and Geometry of the Solutions

Describe the solution set to the *homogeneous* linear system with GaussianElimination(A)

$$\left[\begin{array}{c|c} & 0 \\ A & 0 \\ & 0 \end{array}\right] \xrightarrow{\text{strict Gaussian}} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \\ \end{array} \left[\begin{array}{ccccc} 3 & 0 & -3 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- check for consistency
- if consistent, identify free variables, if any, from variables missing pivots. Parameterically, $x_3 = s$ and $x_4 = t$. Geometrically, two free variables is a plane

Describe the Algebra and Geometry of the Solutions

Describe the solution set to the *homogeneous* linear system with GaussianElimination(A)

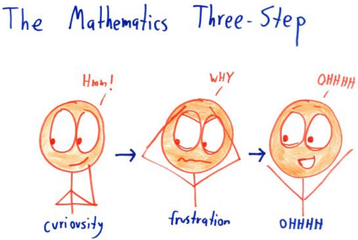
$$\left[\begin{array}{c|c} A & \begin{matrix}0\\0\\0\end{matrix}\end{array}\right] \xrightarrow{\text{strict Gaussian}} \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & = \end{matrix} \\ \left[\begin{array}{ccccc} 3 & 0 & -3 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

- check for consistency
- if consistent, identify free variables, if any, from variables missing pivots. Parameterically, $x_3 = s$ and $x_4 = t$. Geometrically, two free variables is a plane
- back substitute to solve for variables attached to pivots, working from bottom up
  row 2 eq: $0x_1 + x_2 - x_3 + 0x_4 = 0$ so $x_2 = x_3 = s$
  row 1 eq: $3x_1 = 3x_3 - x_4 = 3s - t$ so $x_1 = s - \frac{t}{3}$
- $(x_1, x_2, x_3, x_4) = (s - \frac{t}{3}, s, s, t)$ plane in $\mathbb{R}^4$

Use strict Gaussian on the following augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 1 & 3 & 3 \\ 2 & 2 & h & 4 \end{bmatrix}?$$

a) it takes at least 3 elementary row operations to get to Gaussian here
b) from Gaussian we can see that we have full pivots for all h
c) from Gaussian we can see that some h give us no solutions and a missing pivot
d) more than one of the above is true
e) none of the above



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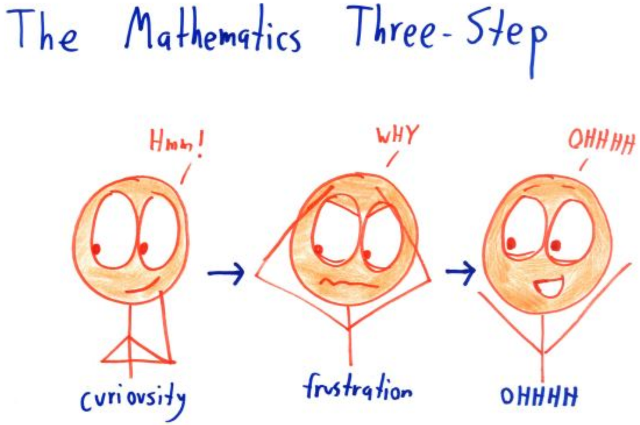
Use strict Gaussian on the following augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 1 & 3 & 3 \\ 2 & 2 & h & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \textcircled{1} & 1 & 0 & 2 \\ 0 & \textcircled{-1} & 3 & -1 \\ 0 & 0 & h & 0 \end{bmatrix}$$ because

$$\xrightarrow{r_2'=-2r_1+r_2} \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2\cdot 1+2=0 & -2\cdot 1+1=-1 & -2\cdot 0+3=3 & -2\cdot 2+3=-1 \\ 2 & 2 & h & 4 \end{bmatrix}$$

$$\xrightarrow{r_3'=-2r_1+r_3} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 3 & -1 \\ -2\cdot 1+2=0 & -2\cdot 1+2=0 & 2\cdot 0+h=h & -2\cdot 2+4=0 \end{bmatrix}$$

no row is [0 0 0 nonzero] so the system is always consistent
1 is a pivot for x, -1 is a pivot for y
when $h \neq 0$, h is a pivot for z giving a unique solution
when $h = 0$, z is missing a pivot but we still have a consistent system, thus giving infinite solutions



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Making CONNECTIONS

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