Linear Models in Management Sciences – LP

Notes by Holly Hirst

Successful manufacturers are very concerned with **maximizing** their profits or **minimizing** their costs. They plan carefully how many of their products to make and how much to sell them for. They have **constraints** on numbers of employees and equipment and storage space for or availability of raw materials to consider as well. This is not done by trial and error; a good company will attack the problem very systematically, often using an approach called **Linear Programming**.

The History of Linear Programming

Linear Programming is a relatively new approach. It grew out of troop supply problems that arose during WWII. With a war on several fronts, the size of the problem of coordinating supplies was daunting. Mathematicians looked for an approach that could make use of computers, which were being developed at that time and offered the possibility of doing many simple calculations quickly. In 1947 **George Danzig** developed the simplex algorithm for solving linear programming problems. The simplex algorithm is a simple recipe for solving linear programming problems of any size and is very easy to program on a computer. Since 1947 many other ways of solving linear programming problems have been developed, mostly for problems that have special forms such as those that contain two independent parts.

In recent years linear programming has been applied to problems in almost all areas of human life. A quick check of our library found entire books on linear programming applied to agriculture, city planning, business, and several other areas. Even feed lot operators decide how much of what kinds of food they should give their livestock each day with linear programming. Most computing facilities have software to solve even the largest linear programming problems.

What Makes a Problem Linear?

To solve a linear programming problem we will need to recognize that the problem is "linear" and then formulate a mathematical model. To illustrate what a typical linear programming problem looks like, let's consider the problem:

> *A doll factory wants to plan how many Barbie and Ken dolls to manufacture in a week to maximize company profit. A Barbie doll earns $6.00 in profit and is made of 12 ounces of plastic for her body and 5 ounces of nylon for her hair. A Ken doll earns $6.50 and is made of 14 ounces of plastic. Each doll goes in a box made of 4 ounces of cardboard. The company can only get one weekly shipment of raw materials, including 100,000 ounces of plastic, 30,000 ounces of nylon and 35,000 ounces of cardboard.*

For a problem to be a linear programming problem it must have the properties listed below. Each situation must be examined to see if assuming these ideas is reasonable. Occasionally the assumptions aren't completely met, but are reasonable approximations to reality, so we use the linear programming approach anyway.

Proportionality: In the Barbie and Ken problem it makes sense that twice as much plastic makes twice as many dolls. That's what proportionality means, that it takes proportionally more material to make proportionally more of the product. Profits must also increase in the same proportion as the number of items sold increases. This would be violated if we gave Walmart a quantity discount.
Divisibility: We must be able to produce fractional parts of items. In the Barbie and Ken problem this appears to be violated, because we can't sell two-thirds of a Barbie doll. On the other hand we can interpret making 2/3 of a doll as making two dolls over a three week period, so we can work around this one.
Short Time: Linear programming is a short term tool. Labor constraints aren't forever; we can hire and train new people. Sales constraints can be changed by advertising campaigns. Costs change over time. In the Barbie and Ken problem we are dealing with weekly production figures, which is OK.

These properties are easier to verify than they first appear; linear programming problems will be easy to recognize. Now let's see how to pull the mathematics from the English.

Setting up the Model

The first task is to identify the variables. The Barbie and Ken problem asks us that in the first sentence - "How many Barbie and Ken dolls ...?" so we let:

```
B = the number of Barbies to make per week
K = the number of Kens to make per week
```

The **objective** of a linear programming problem will always be to maximize or minimize a quantity. In the first sentence of the problem we see "maximize company profit." Profit is $6.00 for each Barbie and $6.50 for each Ken, so the total amount of money, Z, that they get will be 6B + 6.50K. We can state this as:

$$\text{Maximize } Z = 6B + 6.5K$$

Now we have to translate all the limiting conditions or **constraints.**

1) **Plastic** is in short supply; the information about plastic says we must use **less than or equal to** 100,000 oz. **12B** is the amount of plastic we use in making Barbies. Similarly, **14K** is the amount of plastic we use in making Kens. So 12B + 14K is the total amount of plastic used, and it must be less than 100,000:

$$12B + 14K \leq 100{,}000$$

2) **Nylon** is used only in Barbie dolls, at 5 oz. per doll. That gives:

$$5B \leq 30{,}000$$

3) **Cardboard** is in short supply just like the plastic. We proceed the same way:

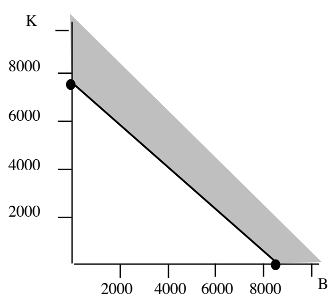
$$4B + 4K \leq 35{,}000$$

The complete mathematical model is below. The last pair of constraints is to remind us that making negative numbers of dolls is not realistic. Even the obvious must be stated in the mathematical formulation.

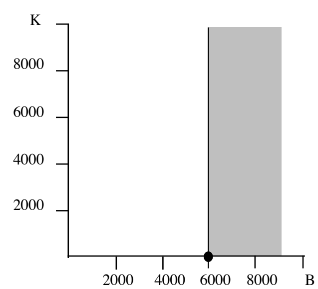
```
B = the number of Barbies to make per week,
K = the number of Kens to make per week.
            Maximize Z =      6B + 6.5K             (profit)
            Subject to:      12B +  14K ≤ 100,000  (plastic)
                              5B        ≤ 30,000   (nylon)
                              4B +   4K ≤ 35,000   (cardboard)
                               B ≥ 0 and K ≥ 0     (non-negativity)
```

So how do we solve this? We need to calculate the (B,K) pair with the largest profit that still fits within the constraints. A point that satisfies all the constraints is called **feasible**. The **feasible region** is the set of points that satisfy all the inequalities. Let's start by sketching the feasible region. (Note that this would be difficult with more than two variables.)

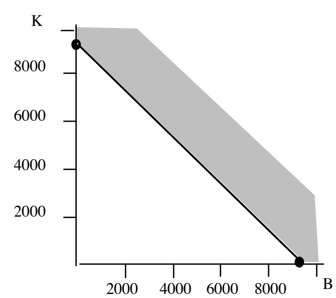
We will graph B ≥ 0 and K ≥ 0 by using only the positive parts of the axes. Now we graph the rest of the inequalities on the same coordinate system, shading the side that **doesn't** work; **the region with no shading will be the feasible region!** Individually:



plastic

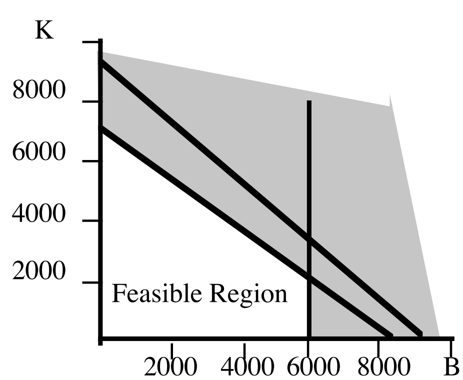


nylon

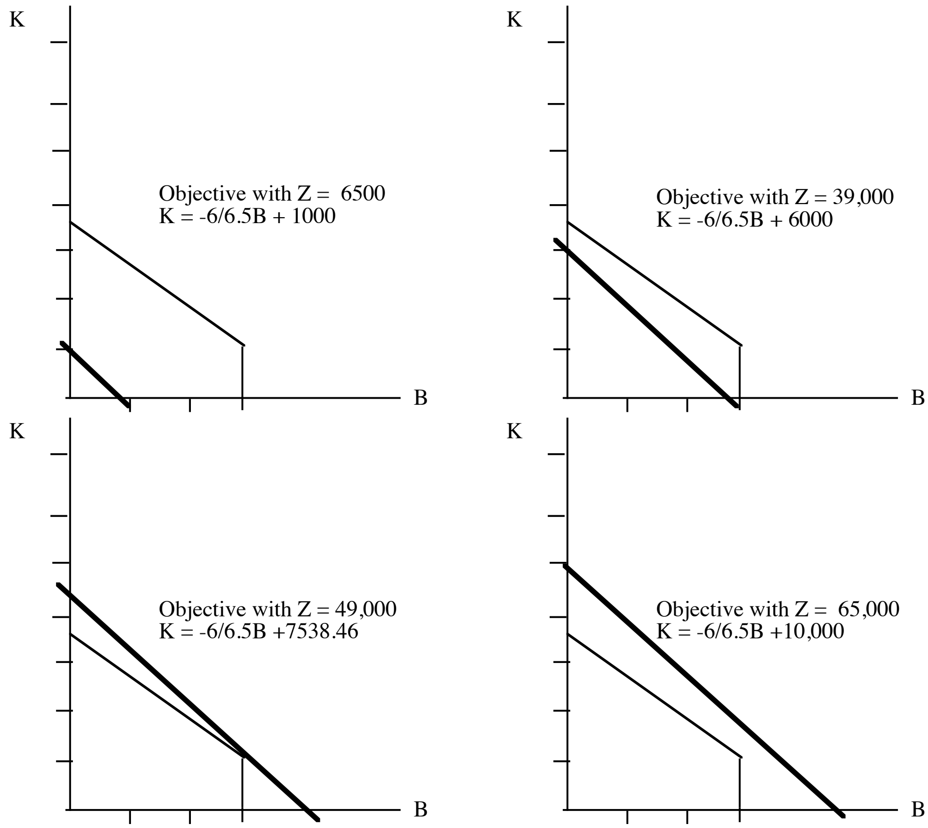


cardboard

Now that we have the feasible region graphed, we want to find the best point, but it is clear that there are too many points to try. We will need some way of choosing the "best" candidates for the maximum. It makes sense that the best answer must occur when some of the constraints are pushed to the limit. What does this mean on the graph?



Let's look at the plastic constraint. There's no room left in this constraint when 12B + 14K **=** 100,000. Graphically, this happens on the line we drew for the plastic constraint. So where on the graph will constraints have no room left? On the border of the region. Exactly which part of the border? We need to use the objective (which is also a line) to test where we leave the feasible region. Consider a few of the graphs below, on which the objective is drawn for various Z-values.



In the first two graphs, a portion of the objective line is within the region. In the third graph, the line has been pushed up until it touches exactly one point of the region (a corner). The fourth graph shows that increasing the objective value more results in the line no longer intersecting the region. So it appears that the objective value can be increased until the objective line intersects the region at a point, namely a corner. This is the fundamental idea for the method we'll use to solve these problems:

The best solution will always occur at a corner point of the feasible region.

If we can find the best point among the corner points we will have the best point in the entire feasible region. That means we have reduced the problem from trying an infinite number of points to only trying the corners, four in this case. Algebraically, a corner point is the intersection of two or more constraint lines, so we use algebra to find the corner points for Barbie and Ken.

Corner Point (B,K)	Profit (6B + 6.5K)
(0,0)	$0
(6000,0)	$36,000
(6000,2000)	$49,000

(0, 7142.86) $46,428.57

Since the third corner point gives the biggest profit, (6000,2000) must give the best or **optimal** solution for all the points. For the Barbie and Ken problem, the very best we can do is to make 6000 Barbie dolls and 2000 Ken dolls for a total profit of $49,000.

Notice that we can immediately see other facts about the solution:

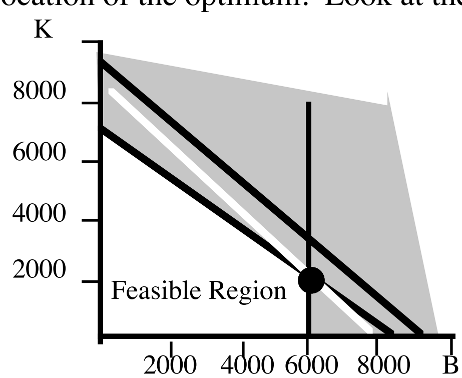
- Since the corner where the solution occurred involved plastic and nylon, there is none of either of these raw materials left over. We refer to plastic and nylon as **binding constraints** and as **scarce resources**, since we use them up with the optimal answer.
- Since cardboard never was a factor, we have leftover regardless of the objective values (i.e., the profit figures for either doll).
- What if the objective was changed so that the profit for the Ken was $6? How much work would we have to do? Just the very last step of checking the profit at each corner. What if one of the important, binding constraints was changed? Then we'd have to rework all corners associated with that constraint.

Other information we can get from the graphical solution

Objective Coefficient Ranges -- We can tell graphically what happens if we ask the following question: How much will the answer change if we change Ken's profit? We can try a few values and draw a conclusion by trial and error, or we can look at it systematically. Let's call Ken's profit "x." So for the original problem x=6.5. This doesn't affect the constraints, and hence the feasible region, but it will affect the profit. Rewrite the profit in slope intercept form:

$$K = \frac{-6}{x}B + \frac{Z}{x}$$

Note that both the slope and the intercept are affected. So what is happening on the picture? The point where the solution occurs is determined by the slope of the objective in relation to the slopes of the constraints. So we need to see how much we can change x in the slope of the objective and not change the location of the optimum. Look at the picture:



The white line is the objective. Notice that if the slope got as flat as the plastic constraint or as tall as the nylon constraint, we would have a different solution. What are the slopes of these two constraints?

plastic slope = -12/14   nylon slope = ± ∞

We will use -∞ for the slope of nylon since making the objective more vertical involves increasing it in magnitude but keeping it negative. So:

$$-\infty \le \frac{-6}{x} \le \frac{-12}{14}$$

Solving for x gives

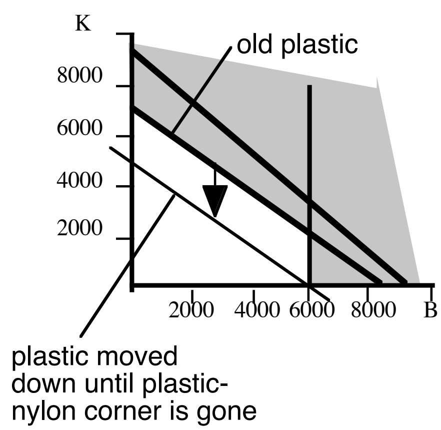
$$0 \le x \le 7$$

which says that the range on profit figures for Ken (which guarantee that the same corner is still the solution) is 0 to 7. Repeating this for Barbie gives a range for her profit figure of 5.57 to infinity.

Right Hand Side Ranges -- We can tell graphically what happens if we ask the following question: How much will the answer change if we change the availability of plastic? We can try a few values and draw a conclusion by trial and error, or we can look at it systematically. Let's call the plastic rhs "x." So for the original problem x=100,000. This affects the constraints, and hence the feasible region. Rewrite the plastic constraint in slope intercept form:

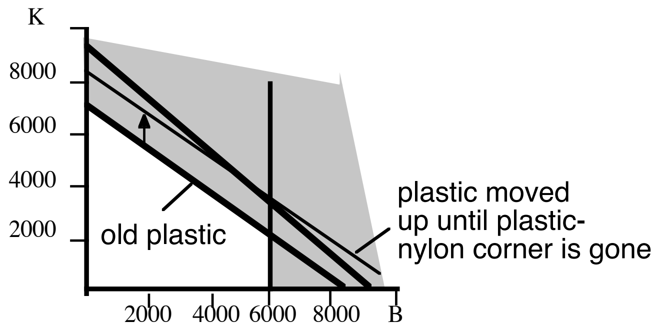
$$K = \frac{-12}{14}B + \frac{x}{14}$$

Here the x doesn't affect the slope of the constraint, just the intercept. Let's look at how far down and up we can slide the plastic constraint before the optimal corner no longer involves where plastic crosses nylon.



When the line is moved down, things change when it is low enough to hit the point (6000,0) on the B axis. So what is x when this happens?

$$0 = \frac{-12}{14}(6000) + \frac{x}{14}$$
$$x = 72{,}000$$



When the line is moved up, things change when it is high enough to hit the point where nylon crosses cardboard. Nylon crosses cardboard at (6000,2750) -- check this by solving the two equations simultaneously! So what is x when this happens?

$$2750 = \frac{-12}{14}(6000) + \frac{x}{14}$$
$$x = 110,500$$

From these two manipulations we see that the rhs value for plastic can range from 72,000 to 110,500, and still allow plastic and nylon to be the binding constraints. Outside of these ranges, other constraints will become binding.

We can perform a similar analysis to see that the rhs range on nylon is 0 to 41666.67 and that the rhs range on cardbaord is 32,000 to infinity.

Minimization Problems

In this example, the feasible region was a closed polygon, so the number of and approximate location of the corner points was immediately obvious from the graph. Sometimes the feasible region seems to be missing a side, as in the example below. This region is referred to as **unbounded**. We can think of the missing corners as being infinite. Such problems can still have solutions.

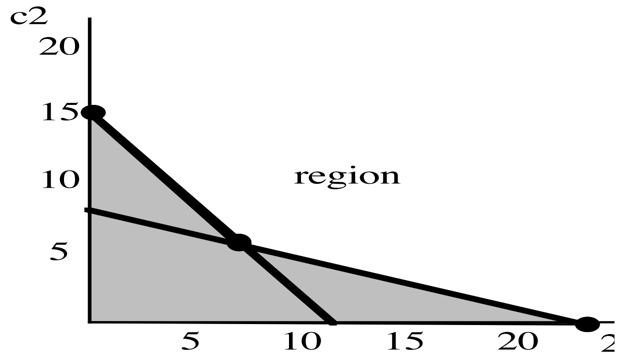
Example: The Watauga county refuse department runs two recycling centers. Center 1 costs $40 to run for an eight hour day. In a typical day 140 pounds of glass and 60 pounds of aluminum are deposited at Center 1. Center 2 costs $50 for an eight hour day, with 100 pounds of glass and 180 pounds of aluminum deposited per day. The county has committed to deliver at least 1540 pounds of glass and 1440 pounds of aluminum per week to encourage a recycler to open up a plant in town. How many days per week should the county open each center to minimize it's cost and still meet the recycler's needs?

<u>Solution:</u> The linear programming translation is as follows.

```
c1 = number of days to open center 1
c2 = number of days to open center 2

minimize    40 c1 +  50 c2 (cost)
subject to
             140 c1 + 100 c2 ≥ 1540 (glass)
```

```
60 c1 + 180 c2 ≥ 1440 (aluminum)
```



The region is unbounded, with three finite corners. Two are easy from the graph: (0, 15.4), (24, 0). The third is found algebraically to be: (6.94, 5.69). Note that the "infinite" corners cannot be best since we're minimizing.

point (c1,c2)	cost = 40 c1 + 50 c2	
(0, 15.4)	$770	
(24,0)	$960	
(6.94,5.69)	$562.10	**BEST**

Solving a two variable linear programming problem graphically is straight-forward. It is also possible to solve a three variable problem graphically, but the corner points become more difficult to locate visually. It is impossible to solve a problem with more than three variables using our graphical method. The corner points of the feasible region must be found somehow using algebra and then checked for the optimal value. The **Simplex** algorithm developed by Danzig solves the problem using this idea.

Problems

1. A store sells 9x12 foot and 9x9 foot rugs, making a profit of $40 and $35 on each, respectively. The factory allows it up to 2000 rugs totaling at most 20,000 square yards annually. How many of each should be sold to maximize profit?

2. A certain area of forest is populated by two species of animals: hawks and snakes. The forest supplies three kinds of food: frogs, mice, and rats. For one year a hawk requires 1 unit of frogs, 2 units of mice, and 2 units of rats, whereas a snake requires 1.2 units of frogs, 1.8 units of mice, and 0.6 units of rats. The forest can normally supply at most 600 units of frogs, 960 units of mice, and 720 units of rats per year. What is the maximum total number of these animals that the forest can support? If a hawk is valued at $150 and a snake is valued at $120, how many animals of each species will maximize the value of the animals?
If there is a wet spring supplied to the forest, then the supply of food becomes at most 720 units of frogs, 960 units of mice, and 600 units of rats. What is then the maximum number of animals that the forest can support? What would happen to the hawks?

3. As part of its campaign to promote its Annual Clearance Sale, the Excelsior Company decided to buy television advertising time on Station KAOS. Excelsior's television advertising budget was $102,000. Morning time costs $3000 per minute, afternoon time costs $1000 per minute, and evening (prime) time costs $12,000 per minute. Because of previous commitments, KAOS could not offer Excelsior more than 6 minutes of prime time or more than a total of 25 minutes of total advertising time over the two weeks in which the commercials were to be run. KAOS estimated that morning commercials would be seen by 200,000 people, afternoon commercials would be seen by 100,000 people, and evening commercials would be seen by 600,000 people. How much morning, afternoon, and evening advertising time should Excelsior buy to maximize exposure of its commercials?

4. A lighting engineer planning to illuminate a store decides at least 30 light fixtures totaling at least 1500 watts are needed. Available are 40-watt and 55 watt fixtures, costing $20 and $27 each, respectively. How many of each should be used to minimize cost?

5. A gardener is mixing up her own fertilizer from 2 brands. Each ounce of brand A contains 5 units if nitrogen, 4 units of phosphorous and 2 units of potassium. Each ounce of brand B contains 4 units of nitrogen, 2 units of phosphorous and 3 units of potassium. The mixture should contain at least 80 units of phosphorous, at least 60 units of potassium and as little nitrogen as possible. How many ounces of each brand should be used?

6. The Boone Weavers make shawls and afghans. They spin yarn, dye yarn and weave yarn for each. A shawl requires 1 hour of spinning, 1 hour of dyeing and 1 hour of weaving. An afghan requires 2 hours of spinning, 1 hour of dyeing and 4 hours of weaving. In a week, there is time to spend at most 13 hours spinning, 10 hours of dyeing and 30 hours weaving. How many shawls and afghans should be made to maximize profit if shawls bring in a profit of $25 and afghans a profit of $40?

7. Joe's Bottlers sells cola and uncola drinks in vending machines. Each machine will hold at most 150 bottles. A bottle of cola brings in $0.25 profit, and a bottle of uncola $0.45. Since more colas than uncolas are sold, they want at least twice as many colas as uncolas, but there should be at least 20 of each kind in a machine. How many of each should be in a machine to maximize profit?

8. An office manager needs to purchase storage lockers. She knows that a regular locker costs $40, requires 4 square feet of floor space, and holds 10 cubic feet of materials. On the other hand, a large storage locker costs $100, requires 8 square feet of floor space, and holds 20 cubic feet. Her budget permits her to spend at most $600, while the office has room for no more than 52 square feet of lockers. The manager desires the greatest cubic feet of storage capacity within the limitations imposed by funds and space. How many of each type of locker should she buy?

9. A dietary manager at a retirement center plans to serve a lunch of roast beef and bread. She is required by law to serve at least 400 calories and 3 milligrams of iron, but would like to minimize her cost. She can buy roast beef for 8¢ per ounce (100 calories and .8 milligrams of iron) and bread for 1¢ per slice (25 calories and .1 milligram iron). Find the amounts of roast beef and bread she should serve.

10. A small generator burns two types of fuel, low sulfur and high sulfur, to produce electricity. In one hour of use each kilogram of low sulfur fuel emits 3 units of sulfur dioxide, generates 3 kilowatts and costs 75 cents, while each kilogram of high sulfur fuel emits 6 units of sulfur dioxide, generates 3 kilowatts and costs 50 cents. The EPA insists that the maximum amount of sulfur dioxide emitted per hour be 9 units. At least 12 kilowatts must be generated per hour. How many kilograms of each fuel should be used hourly to minimize the cost of the fuel used?

Numerical Solution of Linear Programming Problems

Solving a two variable linear programming problem graphically is straight-forward. It is also possible to solve a three variable problem graphically, but the corner points become more difficult to locate visually. It is impossible to solve a problem with more than three variables using our graphical method. The corner points of the feasible region must be found somehow using algebra and then checked for the optimal value. The **Simplex** algorithm developed by Danzig solves the problem using this idea.

The simplex method starts with any **corner** of the feasible region, usually the origin, and then tests to see which edge points towards the neighboring corner with the biggest change for the better in the objective. Then the method "walks" along that edge to the new corner. At the new corner the edges are tested again. If none give a change for the better in the objective, this new corner is the answer, otherwise the method walks and tests again. In this manner the simplex method walks along the edges to the best corner.

The Algebra of the Simplex Method

The main ideas for the Simplex method come from linear algebra. To start we put the problem in standard form by transforming them into linear equalities by adding variables to allow for whatever slack occurs in the inequalities, appropriately called **slack** variables. Here's Barbie and Ken:

```
Z -  6B - 6.5K                = 0        (profit)
    12B +  14K + s1           = 100,000  (plastic)
     5B             + s2      = 30,000   (nylon)
     4B +   4K           + s3 = 35,000   (cardboard)
```

Now we have linear constraints with fewer equations than variables. (3 equations and 5 variables). There are infinitely many solutions to this system. We will algebraically look for the one that maximizes Z. We mentioned before that we start at some obvious corner of the feasible region, usually (0,0). Algebraically this corresponds to setting B = 0 and K = 0 in the equations above. What is the result? We have reduced our system to four equations in four unknowns with solution:

```
s1 = 100000
s2 = 30000
s3 = 35000
```

We will call this the "basic feasible set" corresponding to the corner (0,0). How might we proceed? Let's look at all possible ways of reducing this set of equations to three equations and three unknowns by setting other pairs of variables to zero. Below is a table of results:

Non-Basics	Basics	Feasible?	Objective? (Z=?)
B = 0, K = 0	s1 = 100000, s2 = 30000, s3 = 35000	Yes	0
B = 0, s1 = 0	K = 7142.86, s2 = 30000, s3 = 6428.57	Yes	46428.57
B = 0, s2 = 0	not possible (see constraint 2)	No	xxx
B = 0, s3 = 0	K = 8750, s1 = -22500, s2 = 30000	No (s1 < 0)	xxx
K = 0, s1 = 0	B = 8333.33, s2 = -11666.67, s3 = 1666.67	No (s2 < 0)	xxx
K = 0, s2 = 0	B = 6000, s1= 28000, s3 = 11000	Yes	36000
K = 0, s3 = 0	B = 8750, s1 = -5000, s2 = -13750	No (s2 < 0)	xxx
s1 = 0, s2 = 0	B = 6000, K= 2000, s3 = 3000	Yes	49000
s1 = 0, s3 = 0	B = 11250, K= -2500, s2 = -26250	No (s2< 0)	xxx
s2 = 0, s3 = 0	B = 6000, K = 2750, s1 = -10500	No (s1 < 0)	xxx

We can determine the solution this way, but note that we have essentially found *every* possible intersection of constraints, instead of just the corners of the feasible region. Even with only 3 constraints, this was a lot of work. In general, if we have N constraints and M variables (including slacks) we would have to solve "M choose M-N" systems to solve the problem this way. If N = 5 and M = 9 (for example) we'd have to solve 126 systems of 5 equations and 5 unknowns. (Yuk!) Instead, we'll use our geometric knowledge as follows:

(1) Start at a corner (namely the origin).
(2) Pick the direction (edge) giving the largest increase in the objective
(3) Find the nearest intersection (corner) in that direction
(4) Check to see if we've found the best objective value

Let's explore each of these steps algebraically:

(1) We will use the following tableau to keep track of the coefficients while making calculations:

	B	**K**	**s1**	**s2**	**s3**	**rhs**
OBJ	-6	-6.5	0	0	0	0
s1	12	14	1	0	0	100000
s2	5	0	0	1	0	30000
s3	4	4	0	0	1	35000

We start the method at a feasible point, namely (0,0). This forces the slack variables to have the values 100000, 30000 and 35000, respectively. This is illustrated in the tableau by setting s1 as the "basic" variable for the first constraint. For example, if K = B = 0 the s1 row gives:

```
12(0) +14(0) + s1 + 0(s2) + 0(s3) = 100,000
                s1 = 100,000
```

At any point in the Simplex method, we will keep track of **basic variables**, non-zero variables for which an equation reduces to variable = number in the first column.

(2) This one is easy. We walk in the direction of the largest increase of the objective by

increasing the variable that increases the objective the most (when all of the other variables remain the same). The objective equation looks like:

$$Z - 6B - 6.5K - 0s1 - 0s2 - 0s3 = 0$$
$$Z = 6B + 6.5K + 0s1 + 0s2 + 0s3$$

Since 6.5 is the largest coefficient, increasing K will have the greatest effect. In general, we will increase the variable with the largest coefficient. In the tableau this is the variable with the largest NEGATIVE coefficient. This variable will be called the **entering variable**. Circle the column corresponding to that variable, in this case K.

(3) We know from step two that we're going to walk in the K direction, i.e., along the K-axis. To decide where the nearest corner in that direction is (without having the graph to look at), consider the following idea. If we knew (algebraically) where each constraint crossed the K-axis, the one with the smallest (non-negative) value would be the closest.

How do we find intercepts? Set the other variable(s) equal to zero and solve. In this case we have three constraints:

$$12B + 14K + s1 = 100000,\quad 5B + s2 = 30000,\quad 4B + 4K + s3 = 35000$$
$$\frac{100000}{14} = 7142.86,\qquad \text{undefined},\qquad \frac{35000}{4} = 8750$$

The closest corner is the smallest non-negative number (negatives and undefined would come from infeasible "corners") in this case the first constraint, where s1 becomes 0. This variable will be called the **leaving variable**. Circle this row.

(4) Before we can tell if we're finished we need to find out if increasing any other variables will further increase the objective. We need to update the tableau to reflect that s1 is now 0 and that K has increased before checking. We do this by transforming the existing equations into an equivalent set of equations in which K, s2 and s3 are the basic variables. We build a new tableau with s1 replaced by K, and the coefficient of K in that equation changed to 1 as follows:

	B	**K**	**s1**	**s2**	**s3**	**rhs**
OBJ						
K	12/14	1	1/14	0	0	100000/14
s2						
s3						

We divided both sides of the first constraint by 14 to produce this row. This is referred to as **pivoting**. Now that constraint has K as its basic variable, since B = s1 = 0:

$$12/14\ (0) + K + 1/14\ (0) + 0\ s2 + 0\ s3 = 100000/14$$
$$K = 100000/14$$

The other rows are updated as follows:

We update the objective row so that the coefficient of K is 0 (so that we don't choose K to enter again) by adding the new K row times 6.5 to the old objective row.

	B	K	s1	s2	s3	rhs
OBJ	6.5*12/14 - 6	6.5*1 - 6.5	6.5*1/14+0	0	0	6.5*100000/14
K	12/14	1	1/14	0	0	100000/14
s2						
s3						

We update the other constraints so that they remain basic for their variables, i.e., we transform each equation so that the coefficient of K is 0. We do this in the same way as we transformed the objective row.

The new tableau is:

	B	K	s1	s2	s3	rhs
OBJ	-3/7	0	13/28	0	0	325000/7
K	12/14	1	1/14	0	0	100000/14
s2	5	0	0	1	0	30000
s3	4/7	0	-2/7	0	1	45000/7

Now the K column is a unit vector with the 1 in the K-K location. We will refer to this process as **unitizing** the K column. Now we can finally see if increasing another variable will increase the objective. Sure enough, the coefficient of B is negative so, as we saw in step two, we can increase the objective by increasing B.

Continuing the method: We haven't found the best corner, so we go back to step two and continue.

(2) B is the entering variable, i.e., B becomes basic and we'll walk in the B direction next.

(3) What variable leaves, i.e., if we walk in the B direction which corner do we hit next? From step three we know to examine the intercepts $= \frac{\text{rhs}}{\text{enteringcoef}}$ and find the smallest non-negative one. Our choices are: $\frac{50000/7}{6/7} = 8333.33$, $\frac{30000}{5} = 6000$, and $\frac{45000/7}{4/7} = 11250$. Since the second one is smallest (non-negative) we circle that row (the s1 row) as the leaving row.

(4) Updating the tableau by pivoting and unitizing the B column gives:

	B	K	s1	s2	s3	rhs
OBJ	0	0	13/28	3/35	0	49000
K	0	1	1/14	-6/35	0	2000
B	1	0	0	1/5	0	6000
s3	0	0	-2/7	-4/35	1	3000

We are finished, since there are no negative numbers in the objective row. How do we read the solution? Look at the rhs column: OBJ = 49000, K = 2000, B = 6000 and s3 = 3000. Since s1 and s2 are not basic, they are equal to 0. So the solution (which should be no surprise) is: make 6000 Barbie's and 2000 Ken's for a profit of $49,000 with no plastic or nylon left, and 3000 ounces of cardboard left.

Summary of Simplex Method
(1) Build the initial tableau, adding slack variables for each constraint and rearranging the objective.
(2) Choose the entering variable by finding the largest negative coefficient in the objective. Circle that column.
(3) Choose the leaving variable by finding the smallest non-negative $\frac{\text{rhs}}{\text{enteringcoef}}$. Circle that row.
(4) Pivot the tableau by changing the leaving variable to the entering variable in the first column and then unitizing the entering variable column.
(5) If there is still a negative in the objective row, go to step 2, if not the optimal corner has been found.

Shadow Prices

We get one more important piece of information from the final tableau. Consider the final version of the objective:

```
Z + 0 B + 0 K + 13/28 s1 + 3/35 s2 + 0 s3 = 49000
          Z + 13/28 s1 + 3/35 s2 = 49000
          Z + 0.46 s1 + 0.09 s2 = 49000
```

Currently, s1 = 0 and s2 = 0. But if s1 were increased to 1 then to compensate we would also have to add 0.46 to the profit. Likewise if s2 were increased to 1 we would have to add 0.09 to the profit to compensate. What does this say? Since s1 is the amount of plastic left over at optimum (namely 0), we can say that if we get a little more (or less) plastic we would increase (or decrease) the profit by $0.49 for each ounce. Similarly if we get a little more (or less) nylon we would increase (or decrease) the profit by $0.09 for each ounce.

Thus coefficients of the slack variables in the objective row give the **unit worth** of the resource. These numbers are sometimes called **shadow prices** or **dual prices**. We must be a little careful when applying these numbers. They are only good for *small* changes in the amounts of resources. Just how "small?" We know that these prices are non-zero only for constraints related to stuff we run out of -- binding constraints plastic and nylon. And we already know the ranges on the rhs's of the constraints for which the optimal corner will still be where plastic crosses nylon. So the rhs ranges tell us where the shadow prices are good. For example, the plastic shadow price of $0.49 is good for between 72,000 and 110,500 oz of plastic. Let's use this in two examples:

How much will the profit change if we have 80,000 oz of plastic instead of 100,000? 80,000 is in the range, so the shadow price $0.49 per oz is valid. We are getting 20,000 less oz, so we will lose 20,000*.49 = $9800.

How much will the profit change if we have 125,000 oz of plastic instead of 100,000? 120,000 is not in the range, so we cannot tell, since the shadow price is not valid outside the range. To answer this we would have to re-run the problem.

More Problems

11. The Foul Weather Friend Company make 3 kinds of parkas, the polyester filled (PF), the duck-down filled (DF) and the goose-down filled (GF). Each involves 55 minutes of sewing, but it takes 6 minutes to stuff a PF, 8 minutes to stuff a DF and 9 minutes to stuff a GF. The factory has a supplier that will furnish it 200 pounds of duck-down and 135 pounds of goose-down and all the polyester that is needed per week. It takes 11 ounces to fill a jacket. The factory has 650 hours of sewing time available and 80 hours of filling time available per week. It makes $23 on each PF jacket, $33 on each DF and $40 on each GF. How many of each should be made to maximize profit?

12. Boone paint mixes custom colors from 3 base pigments by adding them to white paint. The base pigments are: red, yellow and blue. They have a total of 600 ounces of red pigment, 300 ounces of yellow pigment and 200 ounces of blue pigment available in any given week. The custom colors that they make are:

1) green - sells for $10.00 per gallon - requires 3 ounce blue and 4 ounce yellow
2) purple - sells for $11.00 per gallon - requires 5 ounces of red and 2 ounce of blue
3) pink - sells for $9.00 per gallon - requires 3 ounces red only
4) orange - sells for $8.00 per gallon - requires 4 ounces of yellow and 1 ounce of red

How many gallons of the custom colors should they make in one week to maximize profit given the pigment constraints?

13. The Boone Brewery makes four products called light, dark, ale and premium. These products are made using water, malt hops, and yeast. The brewery has a free supply of water but the amount of the other resources restrict the production. The table below gives the pounds of each resource required to produce one gallon of each product, the pounds of each resource available, and the revenue received for each gallon of product. How much of each product should be made to maximize revenue?

	light	dark	ale	premium	available
malt	1	1	0	3	50 lb
hops	2	1	2	1	150 lb
yeast	1	1	1	4	80 lb
revenue	$6	$5	$3	$7	

14. Swales jeweler's uses rubies and sapphires to produce two types of rings. One requires 2 rubies, 3 sapphires, 1 hour of labor and sells for $400. The second requires 3 rubies, 2 sapphires and 2 hours of labor and sells for $500. Swales has 100 rubies, 120 sapphires and 70 hours of labor available each week. Additional rubies can be purchased at $100 each. Market demand requires that at least 20 of the first and 24 of the second be produced. How many of each type of ring should be produced and how many extra rubies should be bought to maximize revenue?

15. Two warehouses have canned tomatoes on hand and three stores require more tomatoes in stock. TTI, Tomato Transportation Inc., has contracted to deliver these tomatoes. Warehouse I has 100 cases on hand and Warehouse II has 200 cases on hand. The stores which for convenience we will call A, B, and C, have needs of 75 cases, 125 cases and 100 cases respectively. The costs per case for shipping between the warehouses and stores are given in the table below. Find the solution that minimizes the cost of shipping.

	Store A	Store B	Store C
warehouse I	$0.10	$0.14	$0.30
warehouse II	$0.12	$0.20	$0.17

16. The Manteo Metals Corporation wishes to branch out into production and sale of a new metal alloy containing lead, zinc, copper and tin. Tin is a by product of another process and thus is essentially free. The alloys containing lead, zinc and copper available to buy are A, B, C, D, and E. The percent of lead, zinc and copper in each and in the desired alloy are:

	A	B	C	D	E	needed
% lead	10	10	40	60	30	exactly 30
% zinc	10	30	50	30	40	exactly 30
%copper	80	60	10	10	30	exactly 40
cost/lb	4.1	4.3	5.8	6.0	7.5	

What is the best combination of the alloys A, B, C, D, and E giving the needed percentages and minimizing cost?

17. A farmer is raising pigs for market, and he wishes to determine the quantities of the available types of feed that should be given to each pig to meet certain nutritional requirements at a minimum cost. The numbers of each type of basic nutritional ingredient contained within a pound of each feed type is given in the following table, along with the daily nutritional requirements and feed costs:

Nutritional Ingredient	Pound of Corn	Pound of Tankage	Pount of Alfalfa	Minimum Daily Requirement
Carbohydrates	9	2	4	20
Protein	3	8	6	18
Vitamins	1	2	6	15
Costs	7¢	6¢	5¢	

18. A furniture company manufactures 4 kinds of desks. Each desk is first constructed in the carpentry shop and is then send to the finishing shop where it is varnished, waxed and polished. The number of man hours of labor required is as follows:

desk 1	4 hours carpentry	1 hour finishing
desk 2	9 hours carpentry	1 hour finishing
desk 3	7 hours carpentry	3 hours finishing
desk 4	10 hours carpentry	40 hours finishing

Because of limitations in capacity of the plant, no more than 6000 man hours can be expected in the carpentry shop and 4000 in the finishing shop in the next 6 months. The profit from the sale of each desk is: $12 for desk 1, $20 for desk 2, $18 for desk 3, $40 for desk 4. Assuming that raw materials and supplies are available in adequate supply and all desks produced are sold, the company wants to determine how many of each type of desk to make to maximize the profit.

After you have managed to solve this problem the company's ace salesman Fred Fasttalk bursts through the door saying, "I didn't mean to jump the gun but a tremendous opportunity presented itself so I wrote up some orders for the desks for next year. Here they are."

Desk	1	2	3	4
# Sold	60	30	10	50

What do you do now?

19. The school board in Centerville would like to establish racially balanced schools. In order to do this, they have decided to bus some students to other areas of the city. The city has two schools located in zones 2 and 5 of the city, and the school children reside in one of six zones. All children residing in a given area who are also assigned to that school may walk to school. The following table reflects the numbers of minorities living in each area and the distance from each area to each of the schools.

	Distance to School		minority	total
Area	2	5	population	population
1	10	20	90	120
2	0	30	110	130
3	20	10	80	120
4	10	20	70	140
5	30	0	40	140
6	20	10	30	150

Although the Centerville Board would ideally like to have each school made up of the same percent of minorities, this could be a very expensive plan. They have, therefore, set as a goal that the percentage of minorities in each school should be within 10% either way of the total percentage of minorities in Centerville. How should busing be set up if each school can handle at most 450 students assuming the cost of busing is directly proportional to the number of child-miles? How much more does this "cost" than the solution without considering minority balancing?

20. Farm I has 400 acres of land and 600 acre-feet of water available. Farm II has 600 acres of land and 900 acre-feet of water. Farm III has 600 acres of land and 375 acre-feet of water. Only sugar beets, maize and cotton are considered for planting next season. The Co-Op has set limits on the total acreage that can be devoted to each of these crops. The table below gives this information as well as the water needs per acre and the net return per acre on the three crops.

crop	max acres	water	net return
sugar beets	600	3	$400/acre
cotton	500	2	$300/acre
maize	325	1	$100/acre

The Co-Op has agreed that each farm will plant the same proportion of irrigable land; in other words if one farm plants ½ of its land, all of them must plant ½. How many acres of each crop should be planted at each farm to maximize the net return to the Co-Op? How much does the political constraint of each farm planting the same proportion of land cost?