Bipartite Graphs, Group Membership, and Linear Algebra

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Graphs and Matrices

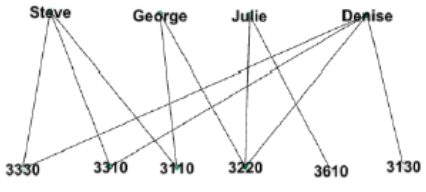
- A ***graph*** is an ordered pair $G = (V;E)$, where
  - V is a set of ***vertices*** (aka nodes)
  - E is a set of ***edges*** (aka links) between the vertices in V.
- A ***matrix*** is an array of numbers grouped into rows and columns
  - $A_{i\,j}$ is the element in row i and column j
  - Matrix multiplication is defined using dot products

    $(AB)_{i\,j}$ = <row i of A, column j of B>
  - A^T is A with rows and columns interchanged

Bipartite graphs and linear algebra combine to answer interesting questions about group membership, using only a little bit of graph theory and matrix multiplication!

Group Membership Networks

- Graphical Representation
  - *V* represents two subsets: individuals and groups
  - *E* represents the connections between individuals and groups
- Example: Students and Classes



- These graphs will be ***undirected, bipartite*** graphs

Reduced Adjacency Matrix

	3330	3310	3110	3220	3610	3130
Steve	1	1	1	0	0	0
George	0	0	1	1	0	0
Julie	0	0	0	1	1	0
Denise	1	1	0	1	0	1

Matrix Multiplication

- $(AA^T)_{ij}$ = dot product of rows i and j
  - Rows represent individuals, and so when two individuals are in the same group the product will be (1)(1)=1 and that instance will be counted. When not in the same group, (0)(0) or (0)(1) = 0.
- $(A^TA)_{ij}$ = dot product of columns i and j
  - Columns represent groups, and so when two groups contain the same individuals that instance will be counted as above.

Person-Person Affiliation Network (AA^T)

- Each row and column represents an individual in the original row order.
- Diagonal entries give total number of groups each individual belongs to.
- Upper triangular entries give the number of groups two individuals have in common.

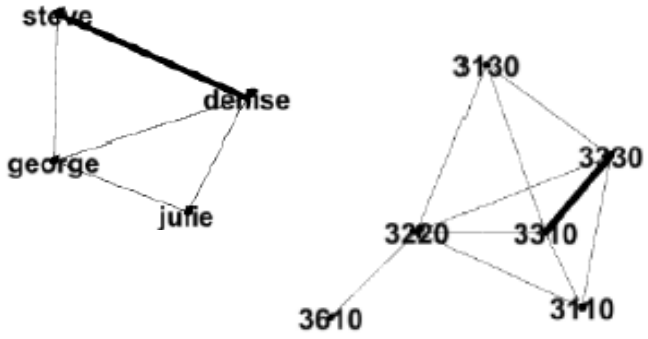
$$AA^T = \begin{bmatrix} 3 & 1 & 0 & 2 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 1 & 1 & 4 \end{bmatrix}$$

Group-Group Affiliation (A^TA)

- Each row and column represents a group in the original column order.
- Diagonal entries give total number of individuals in the group.
- Upper triangular entries give the number of individuals two groups have in common.

$$A^TA = \begin{bmatrix} 2 & 2 & 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Back to Graphs – Now with Weighted Edges



Ex 1: Revolutionary War

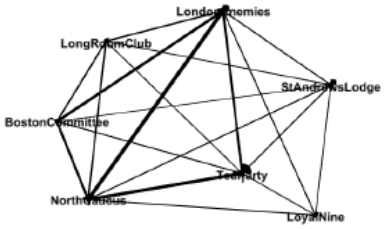
- What role did social clubs play?

- Many patriots were members of social clubs involved in planning acts against the British in Boston.
  - John Adams: North Caucus and the Long Room Clubs.
  - Paul Revere: member of seven clubs.
- Listing of the memberships of seven social clubs: David Hackett Fisher (http://www.amazon.com/Paul-Reveres-David-Hackett-Fischer/dp/0195098315)
- Spreadsheet with memberships: Kieran Healy (https://github.com/kjhealy/revere/blob/master/data/PaulRevereAppD.csv)

The Club-Club Affiliation

- Relationship between social clubs 3 (North Caucus) and 7 (London Enemies) was strongest with club 5 (Tea Party) also playing a politically important role in collaboration.
- Club 2 (Loyal Nine) has the weakest connection with other clubs.

$$A^\top A = \begin{bmatrix} 53 & 2 & 3 & 2 & 3 & 1 & 3 \\ 2 & 10 & 3 & 0 & 2 & 0 & 3 \\ 3 & 3 & 59 & 5 & 13 & 9 & 16 \\ 2 & 0 & 5 & 17 & 2 & 5 & 5 \\ 3 & 2 & 13 & 2 & 97 & 3 & 8 \\ 1 & 0 & 9 & 5 & 3 & 21 & 11 \\ 3 & 3 & 16 & 5 & 8 & 11 & 62 \end{bmatrix}$$



Ex 2: Pollinators and Plants

- What should I plant?

- mid-Atlantic state list:
  - smooth penstemon: butterflies, moths, bees, honeybees
  - butterfly milkweed: bees, butterflies
  - joe pye weed: butterflies, bees, hummingbirds
  - purple coneflower: hummingbirds, bees
  - wild bergamot: moths, hummingbirds, bumblebees
  - field thistle: butterflies, bumblebees, hummingbirds
  - marsh blazing star: butterflies, hummingbirds
  - new england aster: honeybees, hummingbirds, moths
  - sneezeweed: bumblebees, honeybees, wasps
  - wrinkle leaf goldenrod: beetles, moths, wasps
- https://xerces.org/pollinator-conservation/plant-lists/

Ex 3: DarkNet Connections

- Organized crime?

- Theory: Criminal networks require a high degree of organization and specialization.

Vendor Abbreviation	Products
MrH	Cocaine, Cannabis, Stimulants, Hash
PS24	Financial Account Numbers, Pirated Software, IDs and Passports, SIM Cards
SFex	Benzos, Cannabis, Cocaine, Stimulants, Fake Prescriptions
OzV	Pirated Software, Erotica, E-Books, Meth
OzDDi	Cannabis Seeds, Weed, Meth, MDMA
TT	Financial Account Numbers, IDs and Passports, Credit Card and CVV Numbers
PEAK	Mescaline, Stimulants, Meth, Psychedelics
PAMFET	MDMA, Speed, Stimulants, Ecstasy
PFish	Weight Loss Pills, Stimulants, Fake Prescriptions, Ecstasy

- Isah, Neagu, and Trundle, “Bipartite Network Model for Inferring Hidden Ties in Crime Data” (https://arxiv.org/abs/1510.02343)

Additional Resources

- Me: hirsthp@appstate.edu
- My handout and slides: appstate.edu/~hirsthp/
- Original inspiration:
  - CCICADA document (“Who is Really in Charge?”)
  - http://ccicada.org/education/ccicada-education-modules/
- Gephi graph software (open source): gephi.org