Motivating Rate of Change Through Modeling

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Thinking about Rate of Change

The concept of rate of change – either average or instantaneous – occurs at many levels of the curriculum. Some example student learning outcomes:

- "Interpret the rate of change and initial value of a linear function in terms of the situation it models" – Eighth Grade (NC and Common Core)
- "Calculate and interpret the average rate of change of a function" – High School (NC and Common Core)
- "Find a derivative interpreted as an instantaneous rate of change" – AP Calculus (ETS)
- "Analyze growth and decay using absolute and relative change" – Content Learning Outcome for Quantitative Reasoning (New Mathways Project)

Thinking about Modeling

The concept of modeling – starting with a real-life situation, making assumptions, mathematizing, solving, interpreting – occurs at many levels of the curriculum, too. Some example student learning outcomes:

- "Apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" – High School (NC and Common Core)
- "When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data" – All Levels (NC and Common Core)
- "Apply simple mathematical methods to the solution of real-world problems" – Quantitative Reasoning for College Graduates (MAA)

A Simple Model: Rabbits and Asparagus

We have rabbits invading our asparagus patch, and we wish to investigate how the rabbit population affects the asparagus patch over the course of ten weeks during the spring and summer. What assumptions might we make?

1. Rabbits are born, and how many are born depends on how many rabbits there are and also how much asparagus there is.
2. Rabbits die, and how many die will depend on how many rabbits there are to compete for food.
3. Asparagus grows steadily.
4. Rabbits eat asparagus when they encounter some.
5. We know how many rabbits and acres of asparagus we have to start.

Proportionality

Mathematizing the growth and grazing of asparagus and the birth and death of the rabbits.

1. When the number of rabbits **and** the amount of asparagus increase, so do the rabbit births.
2. When the number of rabbits increases, so does the competition for food.
3. Asparagus grows steadily, so the change is some constant amount.
4. When the number of rabbits **and** the amount of asparagus increase, more asparagus gets eaten.

The simplest model to use would be proportionality.

Representing the Rates of Change

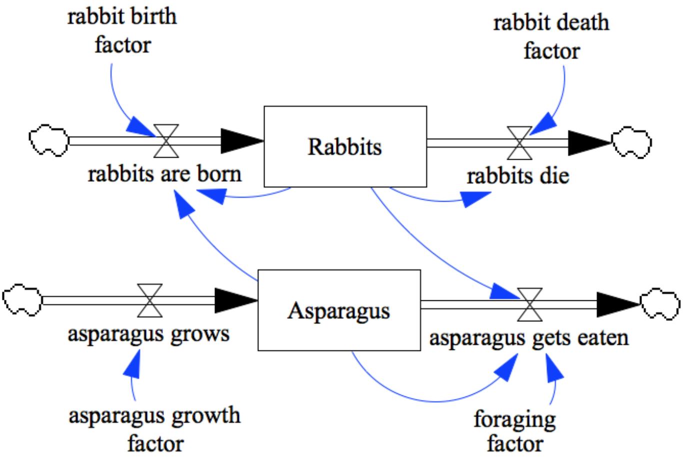
Let $R(t)$ represent the number of rabbits and $A(t)$ represent the amount of asparagus at a particular time t.

1. Rabbit births: $\propto R$ and $\propto A$ imply that the increase due to births: $= rabbit_b \times A \times R$.
2. Rabbit deaths: $\propto R$ implies that the decrease due to competition: $= rabbit_d \times R$.
3. Asparagus growth: constant a_{growth}.
4. Asparagus grazing: $\propto R$ and $\propto A$ imply that the decrease in asparagus due to grazing: $= a_{graze} \times R \times A$

We have 4 growth or death constants, including 3 proportionality factors, in our model.

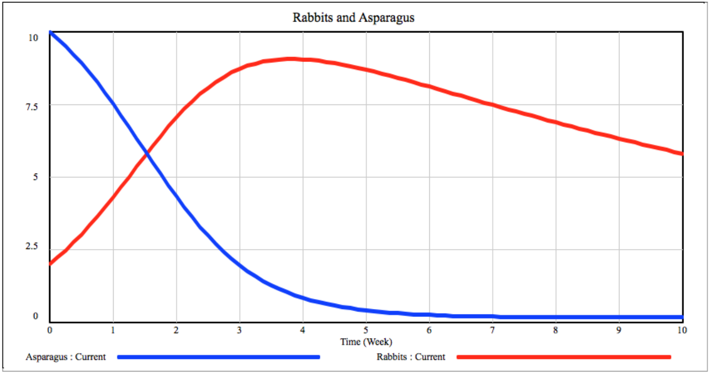
The other model factor: Time step = 2^{-3}.

Diagram - VensimPLE



Results of the Run

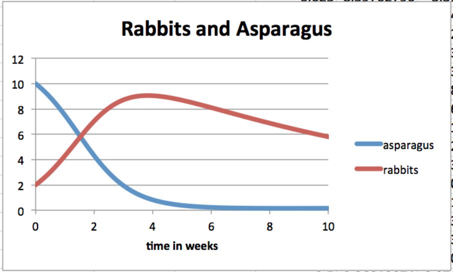
Starting with 10 acres of asparagus and 2 rabbits, and letting all the factors be 0.1...



Does this behavior make sense ?

Excel Model

	A	B	C	D	E	F
1	FACTORS			time	asparagus	rabbits
2	rabbit births	0.1		0	10	2
3	rabbit competition	0.1		0.125	9.7625	2.225
4	asparagus growth	0.1		0.25	9.50348047	2.46870703
5	asparagus grazing	0.1		0.375	9.22271411	2.73111456
6	dt	0.125		0.5	8.9203605	3.01182923
7				0.625	8.59702796	3.3100139
8						434225
9						297153
10						353934
11						318967
12						863115
13						622759
14						121175
15						235512
16						306318
17						058477
18						162204
19						335044
20						350037
21						040184
22				2.5	2.96010074	8.07299225



Adding Coyotes

Coyotes prey on rabbits, so now we have new assumptions:

1. We know the number of coyotes to start.
2. The asparagus growth and foraging are not directly affected.
3. The rabbit deaths are now also proportional to the number of coyotes: $= rabbit_d \times R \times C$.
4. The coyote births are proportional to both the number of coyotes and the number of rabbits: $= coyote_b \times R \times C$.
5. The coyote deaths are proportional to both the number of coyotes: $= coyote_d \times C$.

Other Modeling Scenarios

- Classic Predator Prey
- Spread of Disease *
- Drug Dosage *
- Pollution Flow Through a Chain of Lakes *
- Army Battle
- Producer-Consumer Economy
- Competitive Hunter
- Harvesting in a Shrimp Farm *
- Simple Chemical Reaction *
- and many more..........

Classic Predator-Prey

Lotka and Volterra (c. 1924)

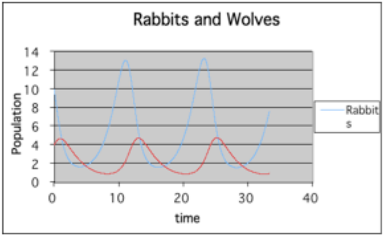
- A prey species grows at a rate proportional to the number of individuals present at any given time.
- Introduce a predator, which will kill prey species at a rate proportional to the number of predator-prey interactions.
- The predator species grows at a rate proportional to the number of predator-prey interactions, as the food source is crucial for reproduction.
- The predator species decreases at a rate proportional to the number of predators, in order to account for competition among the predators.

Classic Predator-Prey - cont'd.

$$\frac{dX}{dt} = rb \cdot X - rd \cdot X \cdot Y$$

$$\frac{dY}{dt} = wb \cdot X \cdot Y - wd \cdot Y$$

time	rabbits	wolves		
0	10	4	dt =	0.01
0.01	9.95	4.0144	rb=	0.7
0.02	9.89982016	4.028691264	rd =	0.3
0.03	9.849468944	4.042871678	wb =	0.08
0.04	9.79895481	4.056939153	wd =	0.44



Resources

A few books....

- Beltrami, E. (1997). *Mathematics for dynamic modeling*, (2nd ed.). San Diego, CA: Academic Press.
- Dossey, J., McCrone, S., Giordano, F., & Weir, M. (2002). *Mathematics Methods and Modeling for Today's Mathematics Classroom*. (Chapter 11). Pacific Grove, CA: Brooks/Cole.
- Fisher, D. (2001). *Lessons in mathematics: A dynamic approach*. Lebanon, NH: ISEEE Systems. Available from iseesystems.com
- Mooney, D., & Swift, R. (1999). *A course in mathematical modeling*. (Chapters 1 and 5). Washington, DC: MAA.

Some websites...

- Shodor Foundation: www.shodor.org
- Maryland Virtual High School: mvhs.shodor.org
- Holly Hirst: mathsci2.appstate.edu/ hph/dynamic/