Proportional Reasoning and Rate of Change Through Modeling

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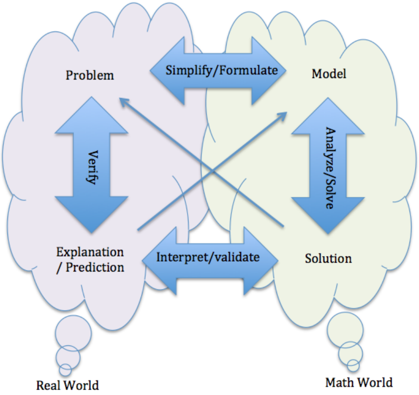
Proportional Reasoning and (Rate of) Change

- “Interpret the rate of change and initial value of a linear function in terms of the situation it models” – Eighth Grade (NC and Common Core)
- “Utilize proportional reasoning to solve contextual problems and make conversions involving various units of measurement; Identify, interpret, and compare linear and exponential rates of growth to make predictions and informed decisions based on data and graphs” – MAT 143 (NCCS Common Course Library)
- “Analyze growth and decay using absolute and relative change” – Content Learning Outcome for Quantitative Reasoning (New Mathways Project)
- “Calculate and interpret the average rate of change of a function” – High School (NC and Common Core)
- “Find a derivative interpreted as an instantaneous rate of change” – AP Calculus (ETS)

Modeling Change



Students can think about change over time in a proportional manner ("the more the merrier"), but need help mathematizing their ideas.

insightmaker.com

A free, web-based tool for modeling that allows users to map out the connections between components of a model, and then input formulas to govern the behavior of the model. From their website:

- System Dynamics: System Dynamics (sometimes called differential equation modeling or dynamical systems modeling) concerns itself with the high-level behavior of a system. It helps you understand the aggregate operations of system on a macro-scale. It is great for cutting away unnecessary detail and focusing on what is truly important in a model.
- Agent Base Modeling: Agent Based models allow you to model individual agents within a system. Where in System Dynamics you might only look at the population as a whole, in Agent Based Modeling you can model each individual in the population and explore the differences and interactions between these individuals.

A Simple Systems Model: Rabbits and Asparagus

We have rabbits invading our asparagus patch, and we wish to investigate how the rabbit population affects the asparagus patch over the course of six weeks during the spring and summer. What assumptions might we make?

Fundamental assumption: The quantities of rabbits and asparagus are changing over time and how much they change is interrelated.

1. Rabbits are born, and how many are born depends on how many rabbits there are and also how much asparagus there is.
2. Rabbits die, and how many die will depend on how many rabbits there are to compete for food.
3. Asparagus grows steadily.
4. Rabbits eat asparagus when they encounter some.
5. We know how many rabbits and acres of asparagus we have to start.

Proportionality

Mathematizing the growth and grazing of asparagus and the birth and death of the rabbits.

1. When the number of rabbits **and** the amount of asparagus increase, so do the rabbit births.
2. When the number of rabbits increases, so does the competition for food.
3. Asparagus grows steadily, so the change is some constant amount.
4. When the number of rabbits **and** the amount of asparagus increase, more asparagus gets eaten.

The simplest model to use would be proportionality.

Representing the Rates of Change

Let $R(t)$ represent the number of rabbits and $A(t)$ represent the amount of asparagus at a particular time t.

1. Rabbit births: $\propto R$ and $\propto A$ imply that the increase due to births: $= rabbit_b \times A \times R$.
2. Rabbit deaths: $\propto R$ implies that the decrease due to competition: $= rabbit_d \times R$.
3. Asparagus growth: constant a_{growth}.
4. Asparagus grazing: $\propto R$ and $\propto A$ imply that the decrease in asparagus due to grazing: $= a_{graze} \times R \times A$

We have 4 growth or death constants, including 3 proportionality factors, in our model.

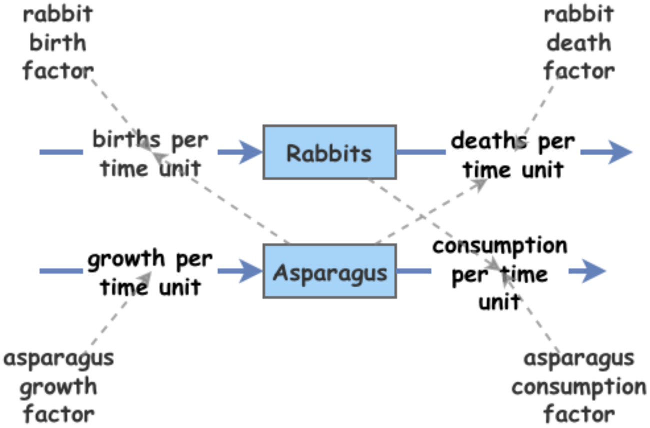
The other important model factor is Time: unit=weeks; duration=0..6; step = 1.

Building the Diagram

- changing quantities are represented by “stocks” - require an initial value
  - Rabbits and Asparagus
- rates of change are represented by “flows” - require directions for how to increase or decrease in each whole unit of time
  - rabbit births and deaths; asparagus growth and consumption
- other model constructs are included as “variables” - these can be constants or more complicated formulas
  - factors representing the proportionality constants
- interdependencies are represented by “links” - linking implies that the item is necessary for the calculation
  - factors and stocks need to be linked to flows

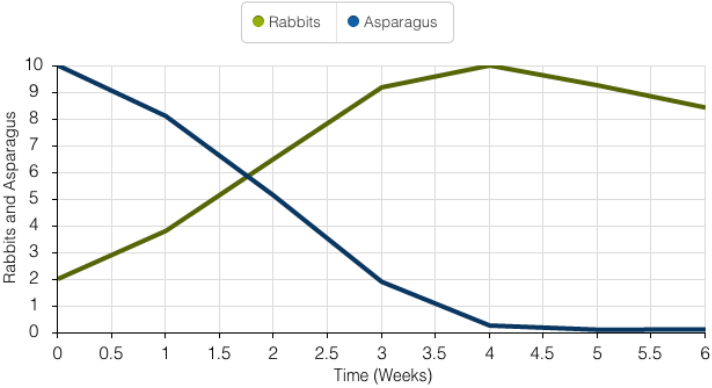
Settings: inputing the simulation length, time units, time step and algorithm

Rabbits and Asparagus



Results of the Run

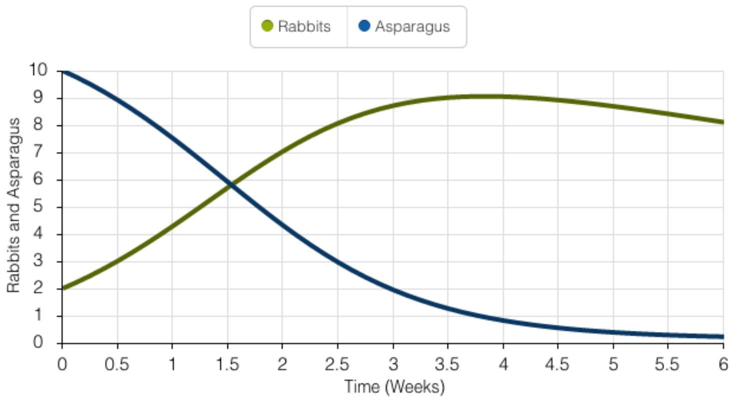
Starting with 10 acres of asparagus and 2 rabbits, and setting all the factors to 0.1.



Does this behavior make sense?

Results of the Run

Time step = 2^{-3} to simulate a more realistic notion of eating,,,

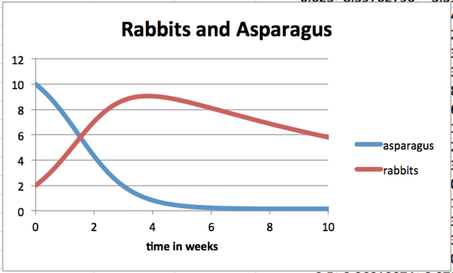


Does this behavior make sense?

Excel Model

It is worth mentioning at this point that these compartmental / systems models can be implemented in a spreadsheet...

	A	B	C	D	E	F
1	FACTORS			time	asparagus	rabbits
2	rabbit births	0.1		0	10	2
3	rabbit competition	0.1		0.125	9.7625	2.225
4	asparagus growth	0.1		0.25	9.50348047	2.46870703
5	asparagus grazing	0.1		0.375	9.22271411	2.73111456
6	dt	0.125		0.5	8.9203605	3.01182923
7				0.625	8.59702796	3.3100139
8						434225
9						297153
10						353934
11						318967
12						863115
13						622759
14						121175
15						235512
16						306318
17						058477
18						162204
19						335044
20						350037
21						040184
22				2.5	2.96010074	8.07299225



Adding Coyotes

Coyotes prey on rabbits, so now we have new assumptions:

1. We know the number of coyotes to start.
2. The asparagus growth and foraging are not directly affected.
3. The rabbit deaths are now also proportional to the number of coyotes: $= rabbit_d \times R \times C$.
4. The coyote births are proportional to both the number of coyotes and the number of rabbits: $= coyote_b \times R \times C$.
5. The coyote deaths are proportional to both the number of coyotes: $= coyote_d \times C$.

Classic Predator-Prey

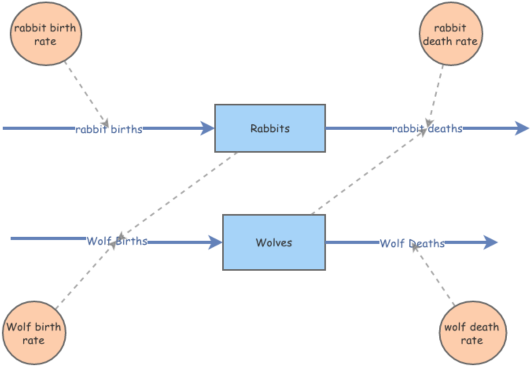
Lotka and Volterra (c. 1924)

- A prey species grows at a rate proportional to the number of individuals present at any given time.
- Introduce a predator, which will kill prey species at a rate proportional to the number of predator-prey interactions.
- The predator species grows at a rate proportional to the number of predator-prey interactions, as the food source is crucial for reproduction.
- The predator species decreases at a rate proportional to the number of predators, in order to account for competition among the predators.

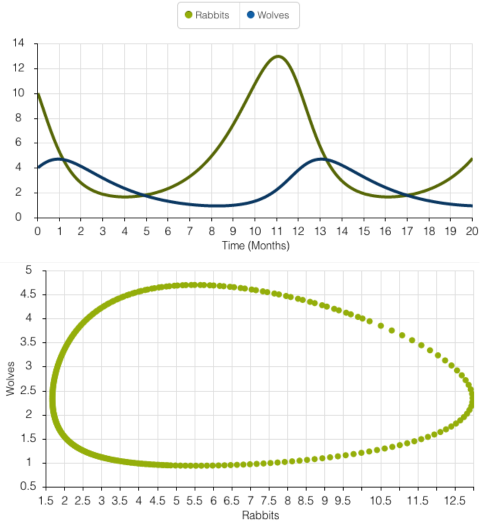
Classic Predator-Prey - cont'd.

$$\frac{dX}{dt} = rb \cdot X - rd \cdot X \cdot Y$$

$$\frac{dY}{dt} = wb \cdot X \cdot Y - wd \cdot Y$$

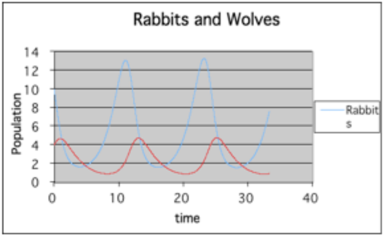


Results of the Run



Predator-Prey - In Excel....

time	rabbits	wolves		
0	10	4	dt =	0.01
0.01	9.95	4.0144	rb=	0.7
0.02	9.89982016	4.028691264	rd =	0.3
0.03	9.849468944	4.042871678	wb =	0.08
0.04	9.79895481	4.056939153	wd =	0.44



Other Modeling Scenarios

- Spread of Disease
- Drug Dosage
- Pollution Flow Through a Chain of Lakes
- Army Battle
- Producer-Consumer Economy
- Competitive Hunter
- Harvesting in a Shrimp Farm
- Simple Chemical Reaction
- and many, many more..........

Resources

A few books....

- Beltrami, E. (1997). *Mathematics for dynamic modeling*, (2nd ed.). San Diego, CA: Academic Press.
- Dossey, J., McCrone, S., Giordano, F., & Weir, M. (2002). *Mathematics Methods and Modeling for Today's Mathematics Classroom*. (Chapter 11). Pacific Grove, CA: Brooks/Cole.
- Fisher, D. (2001). *Lessons in mathematics: A dynamic approach*. Lebanon, NH: ISEEE Systems. Available from iseesystems.com
- Mooney, D., & Swift, R. (1999). *A course in mathematical modeling*. (Chapters 1 and 5). Washington, DC: MAA.

Some websites...

- Systems Thinking Online Text tied to insightmaker: beyondconnectingthedots.com
- Shodor Foundation: shodor.org
- Maryland Virtual High School: mvhs.shodor.org
- Holly Hirst: mathsci2.appstate.edu/ ˜ hph/modeling/