Reverse mathematics, trichotomy, and dichotomy

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Abstract

Using the techniques of reverse mathematics, we analyze the logical strength of statements similar to trichotomy and dichotomy for sequences of reals. Capitalizing on the connection between sequential statements and constructivity, we use computable restrictions of the statements to indicate forms provable in formal systems of weak constructive analysis. Our goal is to demonstrate that reverse mathematics can assist in the formulation of constructive theorems.

1 Axiom systems and encoding reals

We will examine several statements about real numbers and sequences of real numbers in the framework of reverse mathematics and in some formalizations of weak constructive analysis. From reverse mathematics, we will concentrate on the axiom systems RCA_0, WKL_0, and ACA_0, which are described in detail by Simpson [6]. Very roughly, RCA_0 is a subsystem of second order arithmetic incorporating ordered semi-ring axioms, a restricted form of induction, and comprehension for Δ^0_1 definable sets. The axiom system WKL_0 appends

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König's tree lemma restricted to 0-1 trees, and the system ACA_0 appends a comprehension scheme for arithmetically definable sets. ACA_0 is strictly stronger than WKL_0, and WKL_0 is strictly stronger than RCA_0.

We will also make use of several formalizations of subsystems of constructive analysis, all variations of $\mathsf{E\text{-}HA}^\omega$, which is intuitionistic arithmetic (Heyting arithmetic) in all finite types with an extensionality scheme. Unlike the reverse mathematics systems which use classical logic, these constructive systems omit the law of the excluded middle. For example, we will use extensions of $\widehat{\mathsf{E\text{-}HA}}^\omega_\restriction$, a form of Heyting arithmetic with primitive recursion restricted to type 0 objects, and induction restricted to quantifier free formulas. We will often append choice axioms. For example, we will use full choice:

$$\mathsf{AC} : \forall x \exists y A(x, y) \to \exists Y \forall x A(x, Y(x))$$

where x and y may be of any finite type. We will also use quantifier free choice, $\mathsf{QF\text{-}AC}^{0,0}$, which is defined by the same scheme but where x and y are restricted to natural number variables and A is restricted to quantifier free formulas. These systems and many others are treated in detail by Kohlenbach [5]; a short summary appears in the paper of Hirst and Mummert [4].

Both Simpson [6] (following Definition II.4.4) and Kohlenbach [5] (in section 4.1) encode real numbers as rapidly converging sequences of rational numbers. In particular, Simpson defines a real number as a Cauchy sequence of rationals $\alpha = \langle \alpha(k) \rangle_{k \in \mathbb{N}}$ satisfying $\forall k \forall i (|\alpha(k) - \alpha(k+i)| \le 2^{-k})$. Kohlenbach's reals are required to converge slightly more quickly. For rational numbers, equality and inequality can be expressed using quantifier free formulas of arithmetic. The situation for reals is more complicated. Two reals α and β are said to be *equal* if $\forall k (|\alpha(k) - \beta(k)| \le 2^{-k+1})$. Since $\alpha(k)$ and $\beta(k)$ are rationals, this formalization of equality contains only the leading universal quantifier. Because we consider both classical and intuitionistic systems, we need to be expecially careful in defining inequality for reals. For reals α and β, we say $\alpha < \beta$ (or $\beta > \alpha$) if $\exists k (\beta(k) - \alpha(k) > 2^{-k+1})$. We say that $\beta \le \alpha$ if $\neg(\alpha < \beta)$, which is intuitionistically equivalent to $\forall k (\beta(k) - \alpha(k) \le 2^{-k+1})$. Note that $\beta \le \alpha$ is not defined as a disjunction. These definitions are classically equivalent to those following Definition II.4.4 of Simpson [6]; his $\alpha < \beta$ is our $\neg\neg(\alpha < \beta)$. We say α is *positive* if $\alpha > 0$ and *negative* if $\alpha < 0$. Here 0 can be the sequence of zeros, or equivalently any real that is equal to that real. We say α is *non-positive* if α is not positive, which is intuitionistically equivalent to $\alpha \le 0$. Similarly, α is *non-negative* means $0 \le \alpha$. Summarizing, we have the following:

- α is a real means $\forall k \forall i(|\alpha(k) - \alpha(k+i)| \leq 2^{-k})$.
- $\alpha = \beta$ means $\forall k(|\alpha(k) - \beta(k)| \leq 2^{-k+1})$.
- $\alpha < \beta$ means $\exists k(\beta(k) - \alpha(k) > 2^{-k+1})$.
- $\beta \leq \alpha$ means $\neg(\alpha < \beta)$, or equivalently, $\forall k(\beta(k) - \alpha(k) \leq 2^{-k+1})$.
- α is positive means $\alpha > 0$, or equivalently, $\exists k(\alpha(k) > 2^{-k+1})$.
- α is negative means $\alpha < 0$, or equivalently, $\exists k(\alpha(k) < -2^{-k+1})$.
- α is non-positive means $\neg(\alpha > 0)$, which is equivalent to $\alpha \leq 0$, which is equivalent to $\forall k(\alpha(k) \leq 2^{-k+1})$.
- α is non-negative means $\neg(\alpha < 0)$, which is equivalent to $\alpha \geq 0$, which is equivalent to $\forall k(\alpha(k) \geq -2^{-k+1})$.

2 Trichotomy and contractive reals

It is well known that the law of trichotomy for reals is equivalent to Bishop's limited principle of omniscience and therefore not provable constructively. Discussions of this can be found in Bridges and Richman [1] and in Bridges and Vîţă [2]. Here is a version of this fact, formulated in the fashion of a reverse mathematics result.

Theorem 1. ($\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{QF\text{-}AC}^{0,0}$) *The following are equivalent:*

1. LPO *(limited principle of omniscience) If $f : \mathbb{N} \to \{0,1\}$ then either $\exists n(f(n) = 1)$ or $\forall n(f(n) = 0)$.*
2. *If α is a real number and $\alpha \geq 0$, then either $\alpha > 0$ or $\alpha = 0$.*
3. *If α is a real number, then $\alpha < 0$, $\alpha = 0$, or $\alpha > 0$.*

Consequently, none of these statements are provable in $\mathsf{E\text{-}HA}^{\omega} + \mathsf{AC}$.

Proof. We sketch the equivalences, working in $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{QF\text{-}AC}^{0,0}$. First assume LPO and suppose $\alpha \geq 0$. Define f by $f(n) = 1$ if $\alpha(n) > 2^{-n+1}$ and $f(n) = 0$ otherwise. Apply LPO to determine if $\alpha > 0$ or $\alpha = 0$.

Next, assume item 2 and let α be a real. Define reals ρ and σ by

$$\rho(n) = \begin{cases} \alpha(n) & \text{if } \alpha(n) \geq 0, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{and} \qquad \sigma(n) = \begin{cases} -\alpha(n) & \text{if } \alpha(n) \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that ρ and σ satisfy the convergence rate requirements for reals, and that both $\rho \geq 0$ and $\sigma \geq 0$. Apply item 2 to ρ and σ. If $\rho > 0$, then $\alpha > 0$. If $\sigma > 0$, then $\alpha < 0$. If both $\rho = 0$ and $\sigma = 0$, then $\alpha = 0$. Thus, item 3 holds.

Finally, assume item 3 and suppose $f : \mathbb{N} \to \{0, 1\}$. Define α as follows. For each n, let $\alpha(n) = 0$ if $f(t) = 0$ for all $t \leq n$, and let $\alpha(n) = 2^{-t}$ if t is the least number less than or equal to n such that $f(t) = 1$. Then $\alpha \geq 0$. Applying item 3, if $\alpha = 0$ then $\forall n(f(n) = 0)$ and if $\alpha > 0$ then $\exists n(f(n) = 1)$.

The final sentence of the theorem may be derived directly from realizability arguments, or proved by applying a result of Hirst and Mummert [4] as outlined below in Corollary 3. □

Since RCA_0 includes the law of the excluded middle, it proves LPO. Consequently, RCA_0 proves trichotomy for reals, as noted in Section II.4 of Simpson [6]. However, applying trichotomy to a sequence of reals yields a stronger statement.

Theorem 2. (RCA_0) *The following are equivalent:*

1. ACA_0.
2. *If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of non-negative reals, then there is a set $I \subset \mathbb{N}$ such that for all i, $i \in I$ implies $\alpha_i = 0$ and $i \notin I$ implies $\alpha_i > 0$.*

Proof. It is clear that arithmetic comprehension suffices to prove the existence of the set I in item 2, so we only prove the converse.

By Lemma III.1.3 of Simpson [6], we need only show that item 2 suffices to prove the existence of ranges of injections. Let $f : \mathbb{N} \to \mathbb{N}$ be one-to-one. Write $f[s]$ for the finite set $\{f(0), \ldots, f(s-1)\}$. Note that RCA_0 suffices to find the finite set $f[s]$ and, since f is an injection, to calculate $f^{-1}(i)$ given that $i \in f[s]$. Define $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ by

$$\alpha_i(s) = \begin{cases} 2^{-f^{-1}(i)} & \text{when } i \in f[s], \\ 2^{-s} & \text{otherwise.} \end{cases}$$

Straightforward arguments show that $\langle \alpha_i \rangle_{i\in\mathbb{N}}$ is a sequence of non-negative real numbers. Moreover, we see that $\alpha_i > 0$ if and only if i is in the range of f. Thus the range of f exists by applying Δ^0_1 comprehension to find the complement of the set I as provided in item 2. □

The main theorems of Hirst and Mummert [4] show that for certain formulas, if the formula is provable in a constructive setting then $\mathsf{RCA_0}$ proves a related formula for sequences. Since the preceding theorem shows that trichotomy for sequences of reals implies $\mathsf{ACA_0}$ (and so is not provable in $\mathsf{RCA_0}$), we may conclude that trichotomy for single reals is not provable in (an extension of) a constructive axiom system. The axiom system in the following corollary appends an extensionality scheme, full axiom of choice, and independence of premise for ∃-free formulas to Heyting arithmetic. It is a proper extension of a formalization of constructive analysis.

Corollary 3. $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}}$ *does not prove that if* α *is a real and* $\alpha \geq 0$*, then* $\alpha > 0$ *or* $\alpha = 0$*.*

Proof. Apply the contrapositive of Theorem 3.6 of Hirst and Mummert [4] to a formula asserting that for every α there is a natural number n such that if α is a real then either $n = 0$ and $\alpha = 0$, or $n = 1$ and $\alpha > 0$. This formula is in the class Γ_1, so the theorem applies. Theorem 2 shows that $\mathsf{RCA_0}$ does not prove the sequential version, so $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}}$ does not prove trichotomy. □

While the preceding corollary does not provide all the information of Theorem 1, it does indicate a path toward a constructive restriction of trichotomy. In light of Hirst and Mummert's result, a necessary condition for a restriction on α to yield a constructively provable version of trichotomy is that the similarly restricted sequential version of trichotomy is provable in $\mathsf{RCA_0}$. Thus, a restriction of item 2 of Theorem 2 that is provable in $\mathsf{RCA_0}$ may yield a constructively provable form of trichotomy. Of course, Hirst and Mummert's theorem is not biconditional, so the necessary condition is not always sufficient and the correspondence between classical results on sequences and intuitionistic results is not exact. However, work in the reverse mathematics framework can help lead us to constructive results. With this in mind, we consider the following definition.

Definition 4. Let α be a real number. We say that α is *contractive* if for every s and $t > s$, either $\alpha(s) \leq \alpha(t) \leq \alpha(s+1)$ or $\alpha(s+1) \leq \alpha(t) \leq \alpha(s)$.

We say that α is *zero contractive* if it is contractive and for every s and $t > s$, if $\alpha(s) \leq 0$ then $\alpha(t) = 0$.

Informally, each sequential pair of rationals in a contractive real provide upper and lower bounds for the value of the real. Zero contractive reals have the added feature that any occurrence of a non-positive rational indicates that the real is equal to 0. Working in $\mathsf{RCA_0}$, we can prove item 2 of Theorem 2 for sequences of zero contractive reals, as follows.

Theorem 5. ($\mathsf{RCA_0}$) *If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of non-negative zero contractive reals, then there is a set $I \subset \mathbb{N}$ such that for all i, $i \in I$ implies $\alpha_i = 0$ and $i \notin I$ implies $\alpha_i > 0$.*

Proof. Working in $\mathsf{RCA_0}$, suppose $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of non-negative zero contractive reals. Note that if $\alpha_i(0) \leq 0$ or $\alpha_i(1) \leq 0$, then $\alpha_i = 0$. Also, if $\alpha_i(0) > 0$ and $\alpha_i(1) > 0$, then $\alpha_i > 0$. Consequently, the set I defined by $I = \{i \in \mathbb{N} \mid \alpha_i(0) \leq 0 \vee \alpha_i(1) \leq 0\}$ has the desired property and exists by Δ^0_1 comprehension. (The defining formula is actually quantifier free.) $\square$

The preceding leads us to believe that trichotomy for zero contractive reals might be provable constructively. This can easily be verified.

Theorem 6. ($\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction}$) *If $\alpha \geq 0$ is zero contractive, then either $\alpha = 0$ or $\alpha > 0$.*

Proof. Suppose $\alpha \geq 0$ is zero contractive. Note that formulas involving comparisons of specific rational numbers are quantifier free. Thus $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction}$ proves that either $(\alpha(0) \leq 0 \vee \alpha(1) \leq 0)$ or $(\alpha(0) > 0 \wedge \alpha(1) > 0)$. As in the preceding proof, in the first case we have $\alpha = 0$ and in the second case we have $\alpha > 0$. $\square$

One could apply Theorem 3.6 of Hirst and Mummert [4] to Theorem 6 and obtain an alternative proof of Theorem 5. In the next section, we will define k-persistent reals. Theorem 5 and Theorem 6 can also be proved for k-persistent contractive reals by arguments very similar to those above.

3 Dichotomy and persistent reals

Following the pattern of the previous section, we consider Bishop's weak form of the omniscience principle and dichotomy for reals: $\forall \alpha(\alpha \geq 0 \vee \alpha \leq 0)$.

Theorem 7. ($\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{QF\text{-}AC}^{0,0}$) *The following are equivalent:*

1. LLPO *(Lesser limited principle of omniscience) If $f : \mathbb{N} \to \{0, 1\}$ is a function that takes the value 1 at most once, then either $\forall n(f(2n) = 0)$ or $\forall n(f(2n + 1) = 0)$.*

2. *If α is a real number, then $\alpha \geq 0$ or $\alpha \leq 0$.*

Consequently, neither of these statements are provable in $\mathsf{E\text{-}HA}^{\omega} + \mathsf{AC}$.

Proof. We sketch the equivalence working in $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction} + \mathsf{QF\text{-}AC}^{0,0}$. Assume LLPO and suppose α is a real. Compute f by setting its values to 0 as long as $|\alpha(n)| \leq 2^{-n+1}$. If we discover a (least) k such that $|\alpha(k)| > 2^{-k+1}$, let $f(2k) = 1$ if $\alpha(k) > 0$, let $f(2k + 1) = 1$ if $\alpha(k) < 0$, and set all other values of f to 0. Note that if $\forall n(f(2n) = 0)$ then $\alpha \leq 0$, and if $\forall n(f(2n + 1) = 0)$ then $\alpha \geq 0$.

To prove the converse, suppose $f : \mathbb{N} \to \{0, 1\}$ takes the value 1 at most once. Define α by specifying that

- $\alpha(n) = 0$ if $f(t) = 0$ for all $t \leq n$,
- $\alpha(n) = 2^{-t}$ if there is an even $t \leq n$ such that $f(t) = 1$, and
- $\alpha(n) = -2^{-t}$ if there is an odd $t \leq n$ such that $f(t) = 1$.

Note that if $\alpha \geq 0$ then $\forall n(f(2n+1) = 0)$, and if $\alpha \leq 0$ then $\forall n(f(2n) = 0)$.

The last sentence of the theorem is provable by a realizability argument or as in Corollary 9 below. □

Since RCA_0 proves trichotomy for reals, it also proves dichotomy. Unlike trichotomy, where the sequential form is equivalent to ACA_0, the sequential form of dichotomy is equivalent to WKL_0.

Theorem 8. (RCA_0) *The following are equivalent:*

1. WKL_0.

2. *If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of reals, then there is a set $I \subset \mathbb{N}$ such that for all i, $i \in I$ implies $\alpha_i \leq 0$ and $i \notin I$ implies $\alpha_i \geq 0$.*

Proof. Working in RCA_0, item 2 is easily deduced from Σ^0_1-separation, which is provable in WKL_0 as shown in Lemma IV.4.4 of Simpson [6].

The same lemma of Simpson shows that WKL_0 is equivalent to the separation of ranges of injections with disjoint ranges. We use this to deduce WKL_0 from item 2. Let $f, g : \mathbb{N} \to \mathbb{N}$ be one-to-one functions such that $\forall i \forall j (f(i) \neq g(j))$. Using the bracket notation from the proof of Theorem 2, define the sequence $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ by

$$\alpha_i(s) = \begin{cases} 2^{-f^{-1}(i)} & \text{when } i \in f[s], \\ -2^{-g^{-1}(i)} & \text{when } i \in g[s], \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of real numbers. Moreover, we know that $\alpha_i > 0$ if and only if i is in the range of f, and $\alpha_i < 0$ if and only if i is in the range of g. If $I \subseteq \mathbb{N}$ satisfies item 2, then I contains the range of g and excludes the range of f. $\square$

The following corollary follows immediately by a proof that is almost identical to that of Corollary 3.

Corollary 9. $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathrm{ef}}$ *does not prove that if* α *is a real then* $\alpha \geq 0$ *or* $\alpha \leq 0$.

We have arrived at the point where we hope to formulate a restriction of dichotomy that is computably true for sequences of reals. Here is one prospect.

Definition 10. Let α be a real number.

- We say that α is *upper k-persistent* if

$$\forall s (s \geq k \wedge \alpha(s) \geq 0 \to \exists t (t > s \wedge \alpha(t) \geq 0)).$$

- We say that α is *lower k-persistent* if

$$\forall s (s \geq k \wedge \alpha(s) \leq 0 \to \exists t (t > s \wedge \alpha(t) \leq 0)).$$

- We say that α is *k-persistent* if it is both upper and lower k-persistent.

Informally, if a rational sequence is k-persistent, after the first k entries, the tail contains no last non-positive entry and no last non-negative entry. Every positive real has a tail that consists entirely of positive rationals, so if the k^{th} entry of a k-persistent real is non-positive, then the real cannot be positive. Since RCA_0 proves dichotomy, this guarantees that the real is non-positive. Similarly, if the k^{th} entry is non-negative, then the real must be non-negative. Restricting item 2 of Theorem 8 to k-persistent reals does yield a statement provable in RCA_0.

Theorem 11. (RCA_0) *Fix $k \in \mathbb{N}$. If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of k-persistent reals, then there is a set $I \subset \mathbb{N}$ such that for all i, $i \in I$ implies $\alpha_i \leq 0$ and $i \notin I$ implies $\alpha_i \geq 0$.*

Proof. If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of k-persistent reals, then the set $I \subset \mathbb{N}$ defined by $I = \{i \in \mathbb{N} \mid \alpha_i(k) \leq 0\}$ exists by Δ^0_1 comprehension and satisfies the theorem. □

As in the preceding section, we can verify that the corresponding restricted form of dichotomy is provable in a constructive axiom system.

Theorem 12. ($\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction}$) *If α is k-persistent, then either $\alpha \leq 0$ or $\alpha \geq 0$.*

Proof. If α is k-persistent, consider $\alpha(k)$. Since comparisons of $\alpha(k)$ with 0 are quantifier free formulas, $\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction}$ proves that $\alpha(k) \leq 0$ or that $\alpha(k) \geq 0$. Consider the case when $\alpha(k) \leq 0$. Suppose that there exists a j such that $\alpha(j) - 0 > 2^{-j+1}$. Due to the rate of convergence of α, for all $t > j$, $\alpha(t) > 0$, contradicting the k-persistence of α. Thus $\neg(0 < \alpha)$, which is $\alpha \leq 0$. Similarly, in the case that $\alpha(k) \geq 0$, we have $\alpha \geq 0$. □

4 Variations on persistence

We say that a real is *eventually persistent* if it is k-persistent for some k. RCA_0 proves that every real is eventually persistent, so Theorem 8 holds with sequences of eventually persistent reals substituted in item 2. Examination of the proof of the related computable restriction in Theorem 11 shows that it does not rely on the assumption that every real in the sequence is k-persistent for the same k. We only need to know some value for k for each real. This information can be encoded in an auxiliary function. We say that $h : \mathbb{N} \to \mathbb{N}$ is a *modulus of persistence* for the sequence of reals $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ if for each i, α_i is $h(i)$-persistent.

Theorem 13. (RCA_0) *If h is a modulus of persistence for the sequence of reals $\langle \alpha_i \rangle_{i \in \mathbb{N}}$, then there is a set $I \subset \mathbb{N}$ such that for all i, $i \in I$ implies $\alpha_i \leq 0$ and $i \notin I$ implies $\alpha_i \geq 0$.*

Proof. Use the proof of Theorem 11, replacing k by $h(i)$. □

In the development of real analysis in reverse mathematics [6], many results that require WKL_0 can be proved in RCA_0 for those special cases where functions have a modulus of uniform continuity. In that setting, the existence of the modulus is equivalent to WKL_0. By contrast, the existence of moduli of persistence is unexpectedly strong.

Theorem 14. (RCA_0) *The following are equivalent:*

1. ACA_0.

2. *Every sequence of reals has a modulus of persistence.*

Proof. Given a sequence of reals, a modulus of persistence can be easily be defined by an arithmetical formula using the sequence as a parameter. Thus, ACA_0 proves item 2.

To prove the reversal, we use item 2 to define the range of an injection. Suppose $f : \mathbb{N} \to \mathbb{N}$ is one-to-one. Using the bracket notation from the proof of Theorem 2, define a sequence of reals $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ by

$$\alpha_i(n) = \begin{cases} 2^{-f^{-1}(i)} & \text{if } i \in f[n] \\ 0 & \text{if } i \notin f[n]. \end{cases}$$

Apply item 2 to find a modulus of persistence h for $\langle \alpha_i \rangle_{i \in \mathbb{N}}$. By Δ^0_1 comprehension, the set $X = \{i \mid (\exists k \leq h(i))(f(k) = i)\}$ exists. Clearly, X is a subset of the range of f. Conversely, if for some k we have $f(k) = i$, then for any value j less than k, α_i is not lower j-persistent. Thus, $k \leq h(i)$, and i will be included in X. □

Even though WKL_0 is too weak to prove the existence of a modulus of persistence for a sequence of reals, it does suffice to prove that every sequence of reals is term-wise equal to a sequence that has a modulus of persistence.

Theorem 15. (RCA_0) *The following are equivalent:*

1. WKL_0.

2. *If* $\langle\alpha_i\rangle_{i\in\mathbb{N}}$ *is a sequence of reals, then there is a sequence* $\langle\beta_i\rangle_{i\in\mathbb{N}}$ *of* 0*-persistent reals such that for all* $i \in \mathbb{N}$, $\alpha_i = \beta_i$.

3. *If* $\langle\alpha_i\rangle_{i\in\mathbb{N}}$ *is a sequence of reals, then there is a sequence* $\langle\beta_i\rangle_{i\in\mathbb{N}}$ *of reals with a modulus of persistence such that for all* $i \in \mathbb{N}$, $\alpha_i = \beta_i$.

Proof. To show that $\mathsf{WKL_0}$ proves item 2, suppose $\langle\alpha_i\rangle_{i\in\mathbb{N}}$ is a sequence of reals. Using $\mathsf{WKL_0}$, apply Theorem 8 to find a set I such that for all i, $i \in I$ implies $\alpha_i \leq 0$ and $i \notin I$ implies $\alpha_i \geq 0$. Define β_i as follows. If $i \in I$ and $j \in \mathbb{N}$, let $\beta_i(j) = \alpha_i(j)$ if $\alpha_i(j) < 0$ and let $\beta_i(j) = -2^{-j}$ otherwise. If $i \notin I$ and $j \in \mathbb{N}$, let $\beta_i(j) = \alpha_i(j)$ if $\alpha_i(j) > 0$ and let $\beta_i(j) = 2^{-j}$ otherwise. Straightforward arguments verify that for each i, $\alpha_i = \beta_i$ and that β_i is 0-persistent.

Since the constant 0 function is a modulus of persistence of the sequence $\langle\beta_i\rangle_{i\in\mathbb{N}}$ of item 2, clearly item 2 implies item 3. We complete the proof by deducing $\mathsf{WKL_0}$ from item 3, finding a separating set for the ranges of injections with disjoint ranges. Suppose f and g are one-to-one functions with disjoint ranges. Construct $\langle\alpha_i\rangle_{i\in\mathbb{N}}$ as in the proof of Theorem 8. Apply item 3 to find a term-wise equal sequence of 0-persistent reals, and then apply Theorem 11 to find the set I. As in the proof of Theorem 8, I contains the range of g and excludes the range of f. $\square$

Our choice of the concept of persistence is by no means unique. For example, one can prove item 2 of Theorem 8 in $\mathsf{RCA_0}$ provided that the sequence consists only of non-zero reals. Alternatively, one could consider the following definition. A real α is *sticky* if

- $\forall i((\alpha(i) \geq 0 \wedge \alpha(i+1) \geq 0) \to \alpha(i+2) \geq 0)$,
- $\forall i((\alpha(i) \leq 0 \wedge \alpha(i+1) \leq 0) \to \alpha(i+2) \leq 0)$, and
- $\exists i((\alpha(i) \geq 0 \wedge \alpha(i+1) \geq 0) \vee (\alpha(i) \leq 0 \wedge \alpha(i+1) \leq 0))$.

Intuitively, if α is sticky and we plot pairs $(i, \alpha(i))$ for all i, the graph may alternate above and below the axis for a while, but eventually it will stick above the axis or below the axis. The preceding results all hold with 0-persistence or k-persistence replaced by stickiness; we leave these proofs as exercises for aficionados of sticky things.

5 Relatively persistent reals

Rather than comparing a real to 0 as in the definition of k-persistent, we can compare a pair of reals. If α and β are reals, we say that α and β are *relatively k-persistent* if $\alpha - \beta$ is k-persistent. Here $\alpha - \beta$ is shorthand for $\alpha + (-\beta)$, and addition and unary negation on reals are defined as in Definition II.4.4 of Simpson [6]. Note that $\mathsf{RCA_0}$ proves that $\alpha - \beta$ is k-persistent if and only if $\beta - \alpha$ is k-persistent, so the definition does not depend on the order of the reals. Consider the following result on minima.

Theorem 16. ($\mathsf{RCA_0}$) *The following are equivalent:*

1. $\mathsf{WKL_0}$.
2. *If $\langle\langle \alpha^j_i \rangle_{i \le n_j}\rangle_{j\in\mathbb{N}}$ is a sequence of finite sequences of reals, then there is a function $h : \mathbb{N} \to \mathbb{N}$ such that for every $j \in \mathbb{N}$, $\alpha^j_{h(j)}$ is a minimum for $\langle \alpha^j_i \rangle_{i\le n_j}$. That is, $\alpha^j_{h(j)} \le \alpha^j_i$ for every $i \le n_j$.*
3. *If $\langle(\alpha^j_0, \alpha^j_1)\rangle_{j\in\mathbb{N}}$ is a sequence of ordered pairs of reals, then there is a function $h : \mathbb{N} \to \mathbb{N}$ such that for every $j \in \mathbb{N}$, $\alpha^j_{h(j)}$ is a minimum of (α^j_0, α^j_1).*
4. *If $\langle \alpha_i \rangle_{i\in\mathbb{N}}$ is a sequence of reals then there is a function $h : \mathbb{N} \to \mathbb{N}$ such that for every $i \in \mathbb{N}$, $\alpha_{h(i)}$ is a minimum of $\langle \alpha_j \rangle_{j\le i}$.*

Proof. Working in $\mathsf{RCA_0}$, to prove that $\mathsf{WKL_0}$ implies item 2, construct a tree T as follows. Place the sequence σ of length l in T if and only if for each $j < l$ there is no witness below l that $\alpha^j_{\sigma(j)}$ is not the minimum of $\langle \alpha^j_i \rangle_{i\le n_j}$. More precisely, σ is in T if and only if for every $j < l$ there is no $i \le l$ and $m \le n_j$ such that $\alpha^j_{\sigma(j)}(i) - 2^{-i+1} > \alpha^j_m(i)$. $\mathsf{RCA_0}$ can prove that T is infinite and that the labels used in level j of the tree are bounded by n_j. Using $\mathsf{WKL_0}$, apply Lemma IV.1.4 of Simpson [6] to find an infinite path through T. The sequence of nodes in the path gives a sequence of values for a function h that satisfies item 2.

Clearly, item 3 is a special case of 2. To see that item 3 implies $\mathsf{WKL_0}$, we use Theorem 8. let $\langle \alpha_j \rangle_{j\in\mathbb{N}}$ be a sequence of reals and consider the sequence of ordered pairs $\langle(\beta^j_0, \beta^j_1)\rangle_{j\in\mathbb{N}}$ where for all j, $\beta^j_0 = 0$ and $\beta^j_1 = \alpha_j$. Apply item 3 and use Δ^0_1-comprehension to prove the existence of $I = \{i \in \mathbb{N} \mid h(i) = 1\}$. Then I satisfies item 2 of Theorem 8, which is equivalent to $\mathsf{WKL_0}$.

Item 4 is also special case of item 2, with the finite sequences taken as initial segments of a single infinite sequence. The proof that item 4 implies WKL_0 is part of Theorem 2 of Hirst [3]. □

Applying the concept of relatively k-persistent, we can formulate a computable restriction of item 2 of the preceding theorem.

Theorem 17. (RCA_0) *If $\langle\langle\alpha_i^j\rangle_{i\le n_j}\rangle_{j\in\mathbb{N}}$ is a sequence of finite sequences of reals such that each finite sequence is pairwise relatively k-persistent, then there is a function $h:\mathbb{N}\to\mathbb{N}$ such that for every $j\in\mathbb{N}$, $\alpha_{h(j)}^j$ is a minimum of $\langle\alpha_i^j\rangle_{i\le n_j}$.*

Proof. Working in RCA_0, use the following process to compute $h(j)$ from $\langle\alpha_i^j\rangle_{i\le n_j}$. Linearly order the elements of $\langle\alpha_i^j\rangle_{i\le n_j}$ by starting with α_0^j. If $\alpha_0^j,\dots,\alpha_i^j$ have been linearly ordered, place α_{i+1}^j just to the left of the leftmost $\alpha_{i'}^j$ such that $\alpha_{i'}^j(k)-\alpha_{i+1}^j(k)\ge 0$. (By relative k-persistence, $\alpha_{i'}^j\ge\alpha_{i+1}^j$.) If no such i' exists, place α_{i+1}^j on the extreme right. When all n_j elements have been linearly ordered, let $h(j)$ be the index of the leftmost real. □

The process described in the preceding proof can be executed in a weak constructive setting, proving the following result.

Theorem 18. ($\widehat{\mathsf{E\text{-}HA}}^{\omega}_{\restriction}+\mathsf{QF\text{-}AC}^{0,0}$) *Every finite sequence of pairwise relatively k-persistent reals has a minimum.*

The finite linear orderings in the proof of Theorem 17 can be thought of as finite approximations of embeddings of sequences of reals in $\mathbb{Q}$. Consequently, it is natural to rephrase Theorem 16 as follows.

Theorem 19. (RCA_0) *The following are equivalent:*

1. WKL_0.
2. *If $\langle\langle\alpha_i^j\rangle_{i\le n_j}\rangle_{j\in\mathbb{N}}$ is a sequence of finite sequences of reals, then there is a sequence of embeddings $e_j:n_j+1\to\mathbb{N}$ such that for all i and i' less than or equal to n_j, $e_j(i)\le e_j(i')$ implies $\alpha_i^j\le\alpha_{i'}^j$.*
3. *For every sequence of reals $\langle\alpha_i\rangle_{i\in\mathbb{N}}$ there is an embedding $e:\mathbb{N}\to\mathbb{Q}$ such that for all i and j, $e(i)\le e(j)$ implies $\alpha_i\le\alpha_j$.*

Proof. To prove that WKL_0 proves item 2, use WKL_0 and apply Theorem 15 to find 0-persistent forms of all the pairwise differences of the reals from each finite sequence, acquiring essentially the same information as if each pair was relatively 0-persistent. Use the construction of Theorem 17 to find the linear orderings.

To prove that item 2 implies item 3, apply item 2 to the sequence of initial segments of $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ and find linear orderings for each initial segment. Gradually construct e as follows. Suppose e is defined on $\alpha_0, \ldots, \alpha_j$. To simplify notation, renumber the indices so that the embedded order matches the order of the indices. If the finite ordering provided by item 2 on $\alpha_0, \ldots, \alpha_j, \alpha_{j+1}$ indicates that $\alpha_{j+1} \leq \alpha_0$, let $e(j+1) = e(0) - 1$. If it indicates that $\alpha_j \leq \alpha_{j+1}$, let $e(j+1) = e(j) + 1$. Otherwise, find the least i such that $\alpha_i \leq \alpha_{j+1} \leq \alpha_{i+1}$ or $\alpha_{i+1} \leq \alpha_{j+1} \leq \alpha_i$, and let $e(j+1) = (e(i) + e(i+1))/2$.

To show that item 3 implies WKL_0, deduce item 2 of Theorem 16 by using the embedding from item 3 to select minima of initial segments of a sequence of reals. □

Following the pattern of Theorem 17 and Theorem 18, we could use relative 0-persistence to formulate computable and constructive analogs of item 2 of Theorem 19. If we require that the embeddings of Theorem 19 preserve equality, the corresponding result acts more like trichotomy and requires ACA_0.

Theorem 20. (RCA_0) *The following are equivalent:*

1. ACA_0.
2. *If* $\langle \langle \alpha_i^j \rangle_{i \leq n_j} \rangle_{j \in \mathbb{N}}$ *is a sequence of finite sequences of reals, then there is a sequence of embeddings* $e_j : n_j + 1 \to \mathbb{N}$ *such that for all* i *and* i' *less than or equal to* n_j, $e_j(i) \leq e_j(i')$ *if and only if* $\alpha_i^j \leq \alpha_{i'}^j$.
3. *For every sequence of reals* $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ *there is an embedding* $e : \mathbb{N} \to \mathbb{Q}$ *such that for all* i *and* j, $e(i) \leq e(j)$ *if and only if* $\alpha_i \leq \alpha_j$.

Sketch of proof. To show that ACA_0 proves item 2, it is easy to show that the desired sequence of embeddings is arithmetically definable. The proof that item 2 implies item 3 can be adapted from the similar proof for Theorem 19. To complete the proof, it suffices to use item 3 to derive item 2 of Theorem 2, since that statement implies ACA_0. Given a sequence of reals $\langle \alpha_i \rangle_{i \in \mathbb{N}}$, define the sequence $\langle \beta_i \rangle_{i \in \mathbb{N}}$ by $\beta_0 = 0$ and $\beta_{i+1} = \alpha_i$ for all i. Applying item 3 to $\langle \beta_i \rangle_{i \in \mathbb{N}}$, we know that $\alpha_i = 0$ if and only if $e(i+1) = e(0)$. □

As a final exercise, the reader could define the notion of relatively zero contractive pairs of reals and formulate and prove computable and constructive restrictions of item 2 of Theorem 20.

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