Reverse Mathematics and Dichotomy

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Motto: Dichotomy is not constructive.

A result familiar to constructivists:

Theorem: $(\widehat{\text{E-HA}}^{\omega}_{\restriction} + \text{QF-AC}^{0,0})$ The following are equivalent:

1. LLPO (Lesser limited principle of omniscience) If $f : \mathbb{N} \to \{0, 1\}$ is a function that takes the value 1 at most once, then either f is 0 on evens, or f is 0 on odds.
2. If α is a real number, then $\alpha \geqslant 0$ or $\alpha \leqslant 0$.

Consequently, neither of these statements are provable in $\text{E-HA}^{\omega} + \text{AC}$.

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- $\widehat{\text{E-HA}}^{\omega}_{\restriction} + \text{QF-AC}^{0,0}$ is a weak fragment of analysis based on intuitionistic predicate calculus.
- A real number is coded by a rapidly converging Cauchy sequence of rationals.
- If $\alpha > 0$, there is a witness. $\alpha \leqslant 0$ means $\neg(\alpha > 0)$.

Motto: Dichotomy is computable, but. . .

Theorem: (RCA_0) If α is a real number, then $\alpha \geqslant 0$ or $\alpha \leqslant 0$.

RCA_0 is a weak fragment of classical analysis that includes ordered semi-ring axioms plus induction for Σ^0_1 formulas plus computable comprehension.

. . . but not uniformly computable.

Theorem: (RCA_0) The following are equivalent:

1. WKL_0 (Infinite 0–1 trees have infinite paths.)
2. If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of reals, then there is a set $I \subset \mathbb{N}$ such that for all i, $i \in I$ implies $\alpha_i \geqslant 0$ and $i \notin I$ implies $\alpha_i \leqslant 0$.

Ideas from the reversal

It suffices to use the statement about sequences of reals to find a separating set for the ranges of injections with disjoint ranges.

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Suppose the injections look like this:

n	0	1	2	3	4	...
$f(n)$	4	9	5	8	1	...
$g(n)$	3	2	7	6	10	...

Then build these reals:

$$\alpha_0 = \langle 0, 0, 0, 0, 0, \dots \rangle$$
$$\alpha_1 = \langle 0, 0, 0, 0, 2^{-4}, 2^{-4}, 2^{-4}, 2^{-4}, \dots \rangle$$
$$\alpha_2 = \langle 0, -2^{-1}, -2^{-1}, -2^{-1}, -2^{-1}, \dots \rangle$$

If I contains indices of non-negative reals and includes all positive reals, then I contains $\{n \mid \alpha_n > 0\}$ and avoids $\{n \mid \alpha_n < 0\}$, and so range$(f) \subset I$ and range$(g) \subset I^c$.

Since RCA_0 proves that sequential dichotomy implies WKL_0, RCA_0 (or even RCA) cannot prove sequential dichotomy.

By a result of Hirst and Mummert [3], since RCA cannot prove sequential dichotomy, $\mathsf{E\text{-}HA}^\omega + \mathsf{AC} + \mathsf{IP}^\omega_{\mathsf{ef}}$ does not prove dichotomy.

(AC is a choice scheme and $\mathsf{IP}^\omega_{\mathsf{ef}}$ is an independence of premise scheme for $\exists$-free formulas.)

The result from [3] is not a biconditional, but a *computable restriction* of sequential dichotomy can indicate a candidate for a *constructive* restriction of dichotomy.

Definition: A real α is persistent if

- $\forall s(\alpha(s) \geqslant 0 \to \exists t(t > s \wedge \alpha(t) \geqslant 0))$
  ... the expansion of α has no last non-negative rational and
- $\forall s(\alpha(s) \leqslant 0 \to \exists t(t > s \wedge \alpha(t) \leqslant 0))$
  ... the expansion of α has no last non-positive rational.

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  ... the expansion of α has no last non-positive rational.

Theorem: (RCA_0) If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of persistent reals, then there is a set $I \subset \mathbb{N}$ such that for all i, $i \in I$ implies $\alpha_i \leqslant 0$ and $i \notin I$ implies $\alpha_i \geqslant 0$.

Theorem: $(\widehat{\text{E-HA}}^{\omega}_{\restriction})$ If α is a persistent real, then $\alpha \geqslant 0$ or $\alpha \leqslant 0$.

Moral: Reverse math can assist in formulating constructive results.

Indices of minima

Theorem: [2] (RCA_0) The following are equivalent:

1. WKL_0.
2. For every sequence of reals $\langle \alpha_i \rangle_{i \in \mathbb{N}}$, there is a function $m : \mathbb{N} \to \mathbb{N}$ such that for each n, $\alpha_{m(n)} = \min\{\alpha_0, \ldots, \alpha_n\}$.

Definition: Reals α and β are *pairwise persistent* if $\alpha - \beta$ is persistent.

Theorem: (RCA_0) If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of pairwise persistent reals, then there is a function m such that for each n, $\alpha_{m(n)} = \min\{\alpha_0, \ldots, \alpha_n\}$.

Theorem: $(\widehat{\mathrm{E\text{-}HA}}{}^{\omega}_{\restriction} + \mathrm{QF\text{-}AC}^{0,0})$ Fix k. Every finite sequence of pairwise persistent reals has a minimum.

Enough dichotomy! What about trichotomy?

Theorem: (RCA_0) The following are equivalent:

1. ACA_0.
2. If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of reals, then there is a set $I \subset \mathbb{N}$ such that $i \in I$ if and only if $\alpha_i = 0$.

Definition: A real α is *contractive* if whenever $i < j$, $\alpha(j)$ is in the interval $[\alpha(i), \alpha(i+1)]$.

Theorem: (RCA_0) If $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ is a sequence of contractive persistent reals, then there is a set $I \subset \mathbb{N}$ such that $i \in I$ if and only if $\alpha_i = 0$.

Theorem: ($\widehat{\text{E-HA}}^{\omega}_{\restriction} + \text{QF-AC}^{0,0}$) If α is a contractive persistent real, then $\alpha < 0$ or $\alpha = 0$ or $\alpha > 0$.

Variations on persistence

Definition: A real α is k-persistent if its tail, starting at k, is a persistent real.

Definition: h is a *modulus of persistence* for $\langle \alpha_i \rangle_{i \in \mathbb{N}}$ if for every i, α_i is $h(i)$-persistent.

Theorem: (RCA_0) ACA_0 is equivalent to "every sequence of reals has a modulus of persistence."

Theorem: (RCA_0) The following are equivalent:

1. WKL_0.
2. Every sequence of reals is component-wise equal to some sequence of 0-persistent reals.
3. Every sequence of reals is component-wise equal to a sequence that has a modulus of persistence.

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