Chapter 10

Trouble in Direct Democracy

Focus Questions

In this chapter, we'll explore the following questions:

- What is a referendum election?
- In a referendum election, what does it mean for a voter's preferences to be separable? What problems can occur when voters' preferences are not separable?
- How can binary preference matrices be used to represent the preferences of voters in referendum elections?
- What options have been proposed for resolving the separability problem? What are the pros and cons of each option?

Warmup 10.1. The administration at Little Valley College (LVC) is facing a serious crisis. Due to an influx of new commuter students, there are now more cars on campus than parking spaces. In an attempt to solve this problem, two proposed solutions are being considered:

Proposal 1. Double the price of a student parking permit (thereby encouraging students to carpool or take the bus).

Proposal 2. Build a new parking garage.

An election will be held to allow LVC's students to decide if either or both of these proposals should be approved. The rules for the election are as follows:

- Voting on the two proposals will be conducted simultaneously; that is, both proposals will appear on the same ballot.
- Each voter must register a vote of *yes* or *no* on each proposal.

- A proposal will pass (i.e., be approved) if and only if a majority of the voters vote *yes* on it.

Dave, Mike, and Pete are three roommates at LVC, and their preferences for the outcome of the election are shown in Table 10.1. In this table, Y denotes passage of a proposal and N denotes failure.

	Outcome of Proposal 1/2		
Rank	Dave	Mike	Pete
1	Y/N	N/Y	N/N
2	N/Y	Y/Y	N/Y
3	Y/Y	Y/N	Y/N
4	N/N	N/N	Y/Y

Table 10.1. Preferences for the LVC parking election

(a) Give a reasonable explanation for each of Dave's, Mike's, and Pete's preferences. That is, try to explain intuitively what views or beliefs might have motivated each of their preferences.

(b) If Dave, Mike, and Pete were the only students who showed up to vote, what would the outcome[1] of the LVC parking election be?

(c) Do you think the result from part (b) is a good outcome? Does it accurately reflect the will of the voters? Why or why not?

The election described in Warmup 10.1 is an example of what is commonly known as a *referendum election*. Referendum elections have become increasingly popular in many countries; in the United States, they are used primarily as a way to give voters a direct voice in certain state and local issues. Because referendum elections bypass the representative bodies that typically decide these issues (for instance, state legislatures and city commissions), they are often hailed by proponents as an effective and efficient way to implement *direct democracy*. In fact, according to political scientists Dean Lacy and Emerson Niou [**33**], "the resurrection of direct democracy through referendums is one of the clear trends of democratic politics."

There is definitely a certain appeal to the idea of direct democracy and its implementation through referendum elections. Many have embraced the argument put forth by economist Brian Beedham [**7**] that "direct democracy ... leaves no ambiguity about the answer to the question: What did the people want?" But perhaps we should not be so quick to jump to this conclusion. After all, as we saw in the LVC parking election, referendum

[1] By the *outcome* of a referendum election, we mean the overall result of voting on all of the proposals. So, for instance, the outcome of an election with three proposals might be $Y/Y/Y$, meaning that all three proposals passed. Or the outcome might be $N/Y/N$, meaning that only the second proposal passed.

elections do not always result in outcomes that truly represent the will of the voters. In fact, in that example, the outcome of the election was the least preferred choice of two thirds of the voters!

This isn't the first time we've seen an election lead to a strange or paradoxical outcome, but it is the first example we've seen in the context of referendum elections. So it makes sense to ask: What is it about referendum elections that allows this undesirable behavior to occur? And what can we do to address these problems? In this chapter, we'll consider these and other interesting questions. By doing so, we'll learn about the surprising complexities of referendum elections—and thus be able to more carefully evaluate the claims made by both advocates and opponents of direct democracy.

Even More Trouble

The example we looked at in Warmup 10.1 was interesting, but not as interesting as it could have been. In fact, if we modify the situation slightly, we can see that the outcome could have actually been much worse.

Question 10.2. Consider again the LVC parking election from Warmup 10.1.

(a) Suppose Dave and Mike each recruit 10 of their friends to vote the same way they do. Assuming Pete's preferences remain as they were, would the addition of these 20 extra voters change the outcome of the election?

(b) Suppose Dave and Mike each recruit 100 of their friends to vote the same way they do. Again assuming Pete's preferences remain as they were, would the addition of these 200 extra voters change the outcome of the election?

(c) Liah, the president of the LVC math club, makes the following claim:

> *Even if all 25,461 students at Little Valley College showed up to vote in the parking election, it would be possible for the outcome to be the least preferred choice of all but one of the voters.*

Is Liah correct? Give a convincing argument or example to justify your answer.

Question 10.3. In an effort to improve their living conditions, Dave, Mike, and Pete have pooled their money and are preparing to make some upgrades to their apartment. Each of them suggests a single purchase: Dave wants a new cool-touch George Foreman grill, Mike wants a foosball table, and Pete wants faster internet service so he can video chat with his girlfriend who is studying abroad in Spain. All three of the roommates like each of these suggestions, but they also all secretly believe that they do not have enough money between them to pay for all three. To decide what purchases to make, Pete, inspired by his inordinately powerful role in the parking election, suggests that the matter be settled by a referendum election. He calls for a simultaneous vote on the following three proposals, with each proposal passing if a majority of the voters vote *yes* on it:

Proposal D. Purchase a cool-touch George Foreman grill.

Proposal M. Purchase a foosball table.

Proposal P. Purchase faster internet service.

After the ballots are cast, the three roommates ask their friend Eric to tabulate the results. They wait anxiously until Eric finally returns and announces the result: all three proposals passed!

(a) Give examples of preferences for Dave, Mike, and Pete that would yield this outcome.

(b) Explain how it is conceivable that the outcome of the election could be the least preferred choice of all three of the voters.

(c) Hearing the three roommates' plight, Liah makes another bizarre claim:

> *In a referendum election with an arbitrarily large number of voters, it would be possible for the outcome to be the least preferred choice of all of the voters.*

Is Liah correct this time? Give a convincing argument or example to justify your answer.

(d) What do you think is causing the strange outcomes we've seen in the last two questions? Explain.

The Separability Problem

As we've seen in the last few questions, referendum elections can sometimes produce outcomes that fail to accurately reflect the preferences of the voters. But why? What causes this undesirable behavior? The next question will help us identify one possible explanation.

Question 10.4.* Consider again the LVC parking election from Warmup 10.1.

(a) Suppose Dave somehow found out that Mike and Pete were going to vote N/Y and N/N, respectively. Do you think this information might change the way Dave would vote? Why or why not?

(b) Suppose you told Dave that you knew whether Proposal 2 was going to pass or fail, and then asked him whether he wanted Proposal 1 to pass or fail. What do you think he would say? Explain.

(c) Suppose you told Pete that you knew whether Proposal 2 was going to pass or fail, and then asked him whether he wanted Proposal 1 to pass or fail. What do you think he would say? Explain.

(d) In parts (b) and (c), you should have identified a difference between Dave's and Pete's preferences. In light of this difference, are Mike's preferences more like Dave's or Pete's? Clearly explain your answer.

Question 10.4 reveals some important features of voter preferences in referendum elections. These features are central to what some economists and political scientists have called the *separability problem*, which can be summarized as follows:

- In a referendum election, the outcome that a voter wants on one or more proposals might depend on the outcome of other proposals. (For instance, a voter might want Proposal A to pass, but only if Proposal B also passes.)
- Simultaneous voting (voting on all of the proposals at the same time) doesn't allow voters a way to express these kinds of complex preferences. Instead, voters are forced to separate issues that may be linked in their minds.
- Since voters cannot fully express their true preferences, the outcome of the election might not be a good representation of what the voters really want.

In order to more fully understand the separability problem and thus be able to work toward a satisfying solution, we must first understand what it means for a voter's preferences in a referendum election to be separable. The following definition formalizes this idea.

Definition 10.5. Let v be a voter in a referendum election.

- A collection S of one or more proposals in the election is **separable with respect to v** if v's ranking of the possible combinations of outcomes for the proposals in S does not depend on the outcome of any of the proposals not in S.
- The preferences of v are **separable** (or *completely separable*) if every possible collection of one or more proposals is separable with respect to v.

Question 10.6.* Consider again Dave's, Mike's, and Pete's preferences and the proposals in the LVC parking election from Warmup 10.1.

(a) Is Proposal 1 separable with respect to Dave, Mike, or Pete? If so, with respect to which voter(s) is it separable? Explain how you know.

(b) Is Proposal 2 separable with respect to Dave, Mike, or Pete? If so, with respect to which voter(s) is it separable? Explain how you know.

(c) Are any of Dave's, Mike's, or Pete's preferences completely separable? If so, whose? Explain how you know.

Question 10.7. In the example from Question 10.3, are Dave's, Mike's, and Pete's preferences likely to be separable? Why or why not?

Question 10.8. Suppose a voter in a referendum election with three proposals ranks the possible outcomes as follows:

$$Y/Y/Y \succ Y/N/Y \succ Y/N/N \succ Y/Y/N \succ N/N/N \succ N/Y/Y \succ N/Y/N \succ N/N/Y$$

Are this voter's preferences separable? Why or why not?

As you probably observed in Question 10.8, it can be difficult to determine if a voter's preferences in a referendum election are separable. This is primarily due to the fact that the definition of separability requires that we consider every possible collection of proposals. For elections with just two proposals, this task only involves looking at each individual proposal. But for elections with more than two proposals, the situation is not quite so simple.

Question 10.9.* Suppose you wanted to know if a particular voter's preferences in a referendum election were separable.

(a) If there were three proposals in the election, what is the maximum number of collections of proposals you would need to consider?

(b) If there were five proposals in the election, what is the maximum number of collections of proposals you would need to consider?

(c) If there were ten proposals in the election, what is the maximum number of collections of proposals you would need to consider?

(d) In each of parts (a)–(c), why do you think you were asked about the *maximum* number of collections of proposals you would need to consider? Would you ever be able to get by with looking at fewer than this maximum number? Clearly explain your answers.

As we saw in Question 10.9, it can be a lot of work to check to see if a voter's preferences in a referendum election are separable—especially if the election involves a large number of proposals. Fortunately, however, there are some shortcuts, which we'll learn about soon. But first, let's take a few minutes to explore a mathematical model that provides a convenient way to represent voter preferences in referendum elections.

Binary Preference Matrices

For each voter in a referendum election, we can represent the voter's preferences with a rectangular array of zeroes and ones called a **binary preference matrix**. The next question illustrates the correspondence between voter preferences and binary preference matrices.

Question 10.10.* Consider again the LVC parking election from Warmup 10.1. For this election, the binary preference matrices that result from Dave's, Mike's, and Pete's preferences are shown in Table 10.2.

$$\begin{array}{ccc} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \\ \text{Dave} & \text{Mike} & \text{Pete} \end{array}$$

TABLE 10.2. Binary preference matrices for the LVC parking election

Given the preferences in Table 10.1, how do you think the binary preference matrices in Table 10.2 were formed? (Note: If you answer correctly, you will have discovered the general rule for forming and interpreting binary preference matrices.)

Question 10.11. Which of the following arrays of zeros and ones are binary preference matrices? That is, which could be formed from a voter's preferences in a referendum election with two proposals?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Question 10.12. Write the binary preference matrix that results from the preferences in Question 10.8. Which collections of proposals in the election would be separable with respect to a voter who has these preferences? Which collections of proposals would not be separable? Does using a binary preference matrix make it easier to identify which collections of proposals are separable and which are not? Clearly explain your answers.

Testing for Separability

Now that we understand what binary preference matrices are and how they are formed, we're ready to investigate a couple of tools that can help us more easily test whether a voter's preferences in a referendum election are separable.

Tool 1: Symmetry

Definition 10.13.

- The **bitwise complement** of a row in a binary preference matrix is formed by interchanging all of the zeros and ones in the row (i.e., by replacing all of the zeros with ones and all of the ones with zeros).
- A binary preference matrix is **symmetric** if for every row in the matrix, the ith row from the top is the bitwise complement of the ith row from the bottom.

Question 10.14.* Which of the binary preference matrices in Table 10.2 are symmetric? Which are not? Explain your answers for each one.

Question 10.15. The top half of a symmetric binary preference matrix is shown below. Find its bottom half.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ & \vdots & \end{pmatrix}$$

If we know that the binary preference matrix associated with a voter's preferences in a referendum election is symmetric, we can conclude quite a bit about the voter's preferences. For instance, if we know what the most preferred outcome is, we can easily determine what the least preferred outcome would have to be. And, as we just saw in Question 10.15, if we know the top half of a symmetric binary preference matrix, we can easily determine what the bottom half would have to be.

As you may have noticed in Question 10.14, the only symmetric binary preference matrix in Table 10.2 was the one associated with a voter whose preferences we had previously determined to be separable. This suggests that perhaps a relationship exists between separable preferences and symmetric binary preference matrices. While it would be a mistake to jump to a conclusion about this relationship based only on the examples from Question 10.14, the following theorem confirms our suspicions.[2]

Theorem 10.16. *If a voter's preferences in a referendum election are separable, then the binary preference matrix associated with these preferences will be symmetric.*

[2] Theorem 10.16 is the first of several results in this chapter that we will state but not prove. This doesn't mean that the proofs are extremely difficult or that you couldn't understand them. They simply involve some notation that we haven't used and a slightly more formal (i.e., less intuitive) approach to the idea of separability.

Question 10.17.* What do Theorem 10.16 and your answer to Question 10.12 allow you to conclude about the preferences in Question 10.8?

Question 10.18.* If the binary preference matrix resulting from a voter's preferences in a referendum election is symmetric, must the voter's preferences be separable? Give a convincing argument or example to justify your answer.

So we can see that while the *non*-symmetry of a binary preference matrix can allow us to conclude that a voter's preferences are *not* separable, the symmetry of a binary preference matrix doesn't help us at all if we are trying to show that a voter's preferences *are* separable. For that, we need another tool.

Tool 2: Unions and Intersections

When examining voters' preferences in a referendum election, we might intuitively expect that the separability of certain collections of proposals would be related to the separability of other collections of proposals. For instance, if we knew that Proposal 1 by itself was separable with respect to a particular voter v, and Proposal 2 by itself was also separable with respect to v, we might expect that Proposals 1 and 2 together would be separable with respect to v as well. Is this in fact the case? To find out, let's look at an example.

Question 10.19.* Consider the following binary preference matrix, which represents a voter's preferences in a referendum election with three proposals.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) Are the voter's preferences separable? Why or why not?

(b) Which collections of proposals are separable with respect to the voter? Explain how you know.

(c) Based on your answer to part (b), what can you conclude about the following statement:

> *In a referendum election, if two collections of proposals S and T are separable with respect to some voter's preferences, then the union of S and T (i.e., the collection of proposals belonging to either S or T or both) is also separable with respect to the voter's preferences.*

The observations you made in Question 10.19 may seem a bit counterintuitive at first, so let's take a moment to associate the matrix given in that question with a concrete example. For the sake of this example, imagine that each column represents an ingredient that one could include in a dessert: the first column represents chocolate syrup, the second milk, and the third ice cream. Then each row represents a possible dessert, depending on which ingredients are included. For example, mixing all three ingredients together would make a chocolate milkshake, whereas just chocolate syrup and ice cream would make a sundae. If we view the preference matrix from Question 10.19 in this context, we can see that it is entirely reasonable. Notice the following:

- For each individual ingredient, the preferences of the "voter" are separable. For example, for every possible ice cream/milk combination, the voter always prefers having chocolate syrup to not having it.
- If the voter knows that they'll be having ice cream, then their first choice on the chocolate syrup/milk combination is to have both, and their second choice is to just have chocolate syrup.
- If the voter knows that they *won't* be having ice cream, then their first choice on the chocolate syrup/milk combination is still to have both, but their second choice is to just have milk. (Presumably, even though a glass of milk is a boring dessert, it is more tolerable than a glass of straight chocolate syrup.)
- Because the voter's second choice on the chocolate syrup/milk combination depends on whether they are having ice cream or not, the first two ingredients—though individually separable—are not separable when viewed together.

The above example illustrates one way in which a voter's preferences on the various proposals in an election can depend on each other in complex and nuanced ways. In many cases—for example, if we only care about what a voter's *first* choice is on each possible collection of proposals—then it suffices to check whether or not each individual proposal is separable. In cases where we want to dig a bit deeper, we can take advantage of the following result that deals not with *unions*, but rather *intersections*. If you're not already familiar with this term, the intersection of two sets S and T is denoted $S \cap T$ and is defined to be the set of all elements that belong to *both* S and T.

Theorem 10.20. *If S and T are collections of proposals in a referendum election, and both S and T are separable with respect to a particular voter, then their intersection $S \cap T$ is also separable with respect to the voter.*

Question 10.21.* Suppose that in a referendum election with four proposals, you know that all of the following collections of proposals are separable with respect to some particular voter:

$$\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}, \{\mathrm{A}, \mathrm{C}, \mathrm{D}\}, \{\mathrm{B}, \mathrm{C}, \mathrm{D}\}$$

Which other collections of proposals, if any, would also have to be separable with respect to the voter? Give a convincing argument to justify your answer.

Question 10.22. Suppose you know that in a referendum election with n proposals (where n just represents some arbitrary number), every possible collection of $n-1$ proposals is separable with respect to some particular voter. Which other collections of proposals, if any, would also have to be separable with respect to the voter? Give a convincing argument to justify your answer.

Some Potential Solutions

Now that we've seen how the notion of separability can affect the outcome of referendum elections, we'll conclude this chapter by exploring some of the different strategies that have been proposed for resolving the separability problem. We'll begin with the most obvious solution.

Potential Solution 1: Avoid Nonseparable Preferences

As we saw in our earlier examples, nonseparable preferences can cause all sorts of undesirable and even paradoxical election outcomes. The following theorem is a natural counterpart to this observation.

Theorem 10.23. *In a referendum election in which every voter has separable preferences, a Condorcet winning outcome will be selected whenever one exists.*

Question 10.24.

(a) Does Theorem 10.23 imply that in a referendum election in which every voter has separable preferences, the winning outcome cannot be the least preferred choice of every voter? Explain.

(b) In a referendum election in which every voter has separable preferences, can the winning outcome be the least preferred choice of every voter? Give a convincing argument or example to justify your answer.

Another result worth noting is the following theorem, which concerns avoiding the possibility of manipulation in referendum elections.

Theorem 10.25. *In a referendum election in which every voter has separable preferences, there will never be a situation in which a voter can guarantee a more desirable outcome by voting insincerely (i.e., by voting for an outcome other than their most preferred choice).*

Theorems 10.23 and 10.25 both yield strong, positive conclusions. However, they also have very strong hypotheses. To apply either of these theorems, the preferences of *every single voter* in the election must be separable, meaning that even one instance of nonseparability can nullify the theorems' conclusions. This phenomenon is illustrated in the next question.

Question 10.26. Consider again the LVC parking election from Warmup 10.1, and suppose Dave, Mike, and Pete revise their preferences, resulting in the new binary preference matrices shown in Table 10.3.

$$\underset{\text{Dave}}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}} \qquad \underset{\text{Mike}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}} \qquad \underset{\text{Pete}}{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Table 10.3. Revised binary preference matrices

(a) Which of Dave's, Mike's, and Pete's revised preferences are separable? Which are not separable? Explain how you know.

(b) According to these revised binary preference matrices, and assuming Dave, Mike, and Pete are the only voters in the election, is there a Condorcet winning outcome? If so, will this outcome be selected as the overall winner?

(c) Construct an example to show that in a referendum election with an arbitrarily large number of voters, all but one having separable preferences, a Condorcet winning outcome can fail to be selected as the overall winner. (Hint: Use the binary preference matrices in Table 10.3.)

Potential Solution 2: Set-wise Voting

Since the separability problem is a result of asking voters to separate issues that may be linked in their minds, another way to solve the problem would be to simply not ask voters to make this separation. That is, instead of viewing a vote of $Y/Y/N$ as separate votes on three proposals (votes of *yes* on the first two proposals and *no* on the third), we could view it as a single vote for the outcome $Y/Y/N$ on all three proposals together. Even better, if we did this we could also allow voters to register their entire preference ballots, and then just use our favorite method from Chapters 2–5 (such as plurality, the Borda count, instant runoff, approval voting, etc.) to decide the winning outcome. This technique is often called *set-wise* voting.

Question 10.27. Consider again the example from Question 10.3, and suppose Dave's, Mike's, and Pete's preferences for the election result in the binary preference matrices shown in Table 10.4.

$$\underset{\text{Dave}}{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \qquad \underset{\text{Mike}}{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \qquad \underset{\text{Pete}}{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}}$$

TABLE 10.4. Binary preference matrices for the apartment election

(a) Assuming Dave, Mike, and Pete each vote for their most preferred outcome, what outcome would be produced by set-wise voting along with your favorite voting system from Chapters 2–5?

(b) Do you think the outcome under set-wise voting in part (a) more accurately reflects the will of the voters than the outcome under the standard proposal-by-proposal method? Why or why not?

Question 10.28. Write a short discussion of the pros and cons of set-wise voting. Do you think set-wise voting should be used instead of the standard proposal-by-proposal method for all referendum elections? If so, explain why. Otherwise, describe the types of elections for which set-wise voting would be most appropriate.

Potential Solution 3: Sequential Voting

The separability problem ultimately boils down to a lack of information; voters might need information about the outcomes of some of the proposals in a referendum election before they can accurately voice their preferences on other proposals. It would seem natural then to attempt to provide this information to voters by conducting a sequence of elections instead of requiring voters to vote on all of the proposals at the same time. To see how such a method might work, let's look at an example.

Question 10.29.* Consider again the LVC parking election from Warmup 10.1, but suppose that instead of voting on both proposals simultaneously, the election is to be held in the following two phases:

- First, the voters will vote on Proposal 1, and the outcome will be announced.
- Then, in a separate election, the voters will vote on Proposal 2.

Assuming Dave, Mike, and Pete are the only voters, and that their preferences are exactly as shown in Table 10.1, would the outcome of the election under this two-phase sequential system be different from the outcome under simultaneous voting? If so, which method (the sequential system or simultaneous voting) do you think results in an outcome that more accurately reflects the will of the voters? Explain.

As we saw in Question 10.29, multi-phase sequential voting can result in outcomes that are better than those that result from simultaneous voting. But will it always?

Question 10.30. Suppose that in a referendum election with three proposals and three voters, the preferences of the voters are as follows.

- **Voter 1:** $Y/N/Y \succ Y/Y/N \succ Y/Y/Y \succ \cdots \succ Y/N/N$
- **Voter 2:** $Y/Y/N \succ Y/Y/Y \succ Y/N/Y \succ \cdots \succ Y/N/N$
- **Voter 3:** $N/Y/Y \succ Y/N/N \succ Y/Y/N \succ Y/Y/Y \succ \cdots \succ Y/N/Y$

(a) What would the outcome of the election be under simultaneous voting?

(b) Suppose the election is to be held sequentially in two phases, with the voters first voting on Proposal 1, and then, after its outcome is announced, on Proposals 2 and 3 simultaneously. What would the overall outcome of the election be under this two-phase sequential system? Do you think this outcome is better or worse than the outcome from part (a)?

(c) Suppose the election is to be held sequentially in three phases, with the voters first voting on Proposal 1, then on Proposal 2, and then on Proposal 3. What would the overall outcome of the election be under this three-phase sequential system? Do you think this outcome is better or worse than the outcome from part (a)?

(d) Suppose the election is to be held sequentially in two phases, with the voters first voting on Proposals 1 and 2 simultaneously, and then on Proposal 3. What would the overall outcome of the election be under this two-phase sequential system? Do you think this outcome is better or worse than the outcome from part (a)?

Question 10.30 shows (surprisingly!) that sequential voting does not always result in an outcome that is better than the outcome of simultaneous voting. In fact, the outcome under simultaneous voting can actually be more desirable to a majority of voters than the outcome for the same election under sequential voting. As it turns out, the additional information provided by sequential voting can at times be detrimental rather than beneficial. In addition, the outcome of a sequential election can depend on the order in which the voting takes place (much like the outcome of sequential pairwise voting can depend on the agenda), which introduces the potential for manipulation. Finally, sequential voting can be costly and time consuming, particularly for elections with a large number of proposals.

With all of that said, sequential voting can be helpful (or at least not harmful) in one very special case.

Theorem 10.31. *Suppose that in a referendum election, the voters first vote simultaneously on all but one of the proposals (with the outcomes announced), and then on the remaining proposal. Let O be the outcome of the election under this two-phase sequential system, and assume that every voter votes sincerely (i.e., for their most preferred outcome). Then all of the following statements will be true:*

- *It is impossible for the outcome under simultaneous voting to be preferred to O by a majority of voters.*
- *It is impossible for O to be a Condorcet losing outcome.*
- *It is impossible for O to be the least preferred choice of every voter.*

Unfortunately, the scenario described in Theorem 10.31 is the only one in which sequential voting has been shown to be consistently effective solution to the separability problem. And even in that very special case, there are still some problems that must be resolved.

Question 10.32. If you wanted to use the method described in Theorem 10.31 to decide the outcome of a referendum election, how would you decide which proposal should be voted on last? Clearly explain your answer.

Question 10.33. In a referendum election with just two proposals, could the outcome produced by simultaneous voting ever be preferred by a majority of voters to an outcome produced by sequential voting? Why or why not?

Potential Solution 4: Contingent Ballots

Question 10.34. Consider again the LVC parking election from Warmup 10.1, but suppose that each voter is provided with a ballot containing the following questions, each of which must be answered with a vote of *yes* or *no*:

- Should proposal 1 be approved?
- Assuming proposal 1 is approved, should proposal 2 be approved?
- Assuming proposal 1 is not approved, should proposal 2 be approved?

(a) Explain how such a ballot (often called a *contingent ballot*) could be used to mimic sequential voting in a referendum election.

(b) What are some of the advantages and disadvantages of using contingent ballots instead of sequential voting?

(c) What are some of the advantages and disadvantages of using contingent ballots instead of simultaneous voting?

(d) In a referendum election with two proposals, could the outcome produced by simultaneous voting ever be preferred by a majority of voters to an outcome resulting from contingent ballots? Why or why not?

Potential Solution 5: Iterative Voting

One recent proposal to solve the separability problem involves allowing voters to change their ballots as many times as they want during a fixed voting period (say, one week), with the current results of the vote—based on the ballots that have been cast already—announced in real time. This system, called *iterative voting*, gives voters the opportunity to strategically revise their votes if they see that voting for their most preferred outcome is unlikely to yield a favorable result. The winning outcome is determined by the votes at the end of the voting period, regardless of what votes have been cast previously, or how many times voters have changed their ballots. To illustrate how this might work, let's look at a couple of examples.

Question 10.35.* Consider again the LVC parking election from Warmup 10.1, and suppose that Dave, Mike, and Pete agree to use iterative voting to decide the outcome. Initially, suppose each of the roommates votes for their most preferred outcome.

(a) A little while after Dave casts his (first) ballot, he checks the results and sees that Proposals 1 and 2 are both failing, with each receiving 1 yes vote and 2 no votes. Explain how Dave could change his vote to ensure a better outcome for himself.

(b) If Dave made the change you identified in part (a), would either Mike or Pete have an incentive to make subsequent changes to their votes?

(c) What do you think the final outcome of the iterative voting election would be? Is this outcome better or worse than the outcome of simultaneous voting? Explain.

Question 10.36. If the election from Question 10.3 was conducted using iterative voting, what do you think the outcome would be? Clearly explain your reasoning, including which voters might change their ballots and how many changes they might make. (You may want to assume that the voters' preferences are as given in Question 10.27.)

As of this writing, the idea of iterative voting remains purely theoretical; there have been no documented cases of iterative voting actually being used to decide the outcome of a referendum election. However, computer simulations have shown that iterative voting produces results that are often better—and very rarely worse—than those of simultaneous voting.

Question 10.37. What do you think are the advantages and disadvantages of iterative voting? What practical considerations would need to be addressed in order to conduct an election using iterative voting?

Potential Solution 6: To Be Determined

Question 10.38. There is still a lot to learn about the separability problem, and much of the recent research has involved undergraduate students. With that inspiration, suggest a potential solution to the separability problem that is different from those discussed in this chapter. Analyze the pros and cons of your potential solution, and describe the types of situations for which it would be best suited.

Questions for Further Study

Question 10.39. In a referendum election with two proposals and any number of voters, is it possible for the outcome to be the least preferred choice of every voter? Give a convincing argument or example to justify your answer.

Question 10.40.

(a) In a referendum election in which every voter has separable preferences and votes sincerely, is it possible for a Condorcet losing outcome to be selected by simultaneous voting? Give a convincing argument or example to justify your answer.

(b) Repeat part (a), but this time assume that all but one of the voters have separable preferences.

Question 10.41. Suppose that in a referendum election with three proposals, a particular voter's preferences can be described as follows:

- The voter's most preferred outcome is for all three proposals to pass, and the voter's least preferred outcome is for all three to fail.
- The voter prefers any outcome in which two proposals pass over any outcome in which only one proposal passes.

Could this voter's preferences be separable? Could they be nonseparable? Give a convincing argument or example to justify each of your answers.

Question 10.42.

(a) Make a list of every possible binary preference matrix for a referendum election with two proposals.

(b) Which of the binary preference matrices you listed in part (a) are symmetric?

(c) Which of the binary preference matrices you listed in part (a) result from preferences that are separable?

(d) Based on your answers to parts (a)–(c), can you characterize the relationship between separable preferences and symmetric binary preference matrices in referendum elections with two proposals?

Question 10.43.

(a) In a referendum election with two proposals, how many different binary preference matrices are possible? How many are symmetric?

(b) In a referendum election with three proposals, how many different binary preference matrices are possible? How many are symmetric?

(c) In a referendum election with n proposals (where n just represents some arbitrary number), how many different binary preference matrices are possible? How many are symmetric?

(d) Using Theorem 10.16 and your answers to parts (a)–(c), explain why the likelihood of a randomly selected voter in a referendum election having separable preferences will decrease toward zero as the number of proposals increases.

(e) In 1990, a referendum election was held in California that, in addition to a number of local initiatives, contained 28 statewide proposals. How likely do you think it was that any of the voters in the election had nonseparable preferences? Explain.

Question 10.44. Find out the details of the Turkish constitutional referendum election from April 2017, and write a summary of your findings. Include in your summary some information about the reasons for and history of the referendum, and its outcome and aftermath.

Question 10.45. Find out the details of the Maine Ranked Choice Voting Initiative from November 2016, and write a summary of your findings. Include in your summary some information about the reasons for and history of the initiative, and its outcome and aftermath.

Question 10.46. Find out the details of California's Drug Price Standards Initiative from November 2016, and write a summary of your findings. Include in your summary some information about the reasons for and history of the initiative, and its outcome and aftermath.

Question 10.47. Find out the details of the United Kingdom European Union membership referendum election from June 2016, and write a summary of your findings. Include in your summary some information about the reasons for and history of the referendum, and its outcome and aftermath.

Question 10.48. Find out the details of the referendum election on the political status of Puerto Rico from November 2012, and write a summary of your findings. Include in your summary some information about the reasons for and history of the referendum, as well as its outcome and aftermath. Do you think the voters' preferences in the election were separable? Why or why not? How does the outcome of this referendum compare to the outcome of a similar referendum from June 2017?

Question 10.49. In the state of Colorado, a proposal was added to the November 2004 presidential election ballot that could have changed the way the state's electoral votes were allocated in that very same election. Find out the details of this proposal, and write a summary of your findings. Include in your summary a description of the proposed new method for allocating the state's electoral votes, who added the proposal to the ballot, why they did so, and the outcome and aftermath of the referendum. Do you think the voters' preferences on the entire ballot were likely to have been separable? Why or why not?

Question 10.50. Find out the details of a recent referendum election in your state, and write a summary of your findings. Include in your summary a statement of each of the proposals in the election and the outcome. Do you think that some of the voters in the election might have had nonseparable preferences? If so, describe any potential links between the proposals in the election, and explain how these links could have affected the separability of the voters' preferences.

Question 10.51. Suppose a voter's preferences correspond to a symmetric binary preference matrix. Could this voter's preferences be completely nonseparable? In other words, is it possible for every collection of proposals to be nonseparable with respect to such a voter's preferences? Give a convincing argument or example to justify your answer.

Question 10.52. For each of the parts below, find a preference matrix corresponding to a voter in an election with three proposals whose preferences are separable on exactly the sets listed (and no others). If you don't think it is possible to find such a matrix, explain why.

(a) $\{1\}$, $\{2\}$

(b) $\{1, 2\}$, $\{3\}$

(c) $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$

(d) $\{1\}$, $\{1, 2\}$, $\{2, 3\}$, $\{3\}$

Question 10.53. Suppose that the first (or last) two rows of a binary preference matrix are bitwise complements of each other. What can you conclude about the corresponding preferences, and why?

Question 10.54. Call a voter's preferences *monoseparable* if they are separable on each individual proposal, but not necessarily on larger sets of proposals. If a voter has monoseparable preferences, must their corresponding binary preference matrix be symmetric? Why or why not?

Question 10.55. Design a contingent ballot that could be used in a referendum election with three proposals. How many questions would such a ballot need to contain?

Answers to Starred Questions

10.4. (a) This information should change the way Dave would vote. If he voted for his first choice, Y/N, then his last choice, N/N, would be the winning outcome. But if he voted (insincerely) for N/Y (his second choice), then N/Y would be the winning outcome.

(b) He would say that it would depend on whether Proposal 2 was going to pass or fail.

(c) He would say that he would want Proposal 1 to fail regardless of whether Proposal 2 was going to pass or fail.

10.6. (a) Proposal 1 is separable with respect to Pete's preferences, but not Dave's or Mike's. (Can you explain why?)

(b) Proposal 2 is separable with respect to Mike's and Pete's preferences, but not Dave's.

(c) Pete's preferences are separable, but neither Dave's nor Mike's are.

10.9. (b) With five proposals, the maximum number of collections of proposals you would need to consider is $\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4} = 30$. (See Chapter 8 for a description of this notation.)

10.10. In each binary preference matrix, a 1 represents a Y and a 0 represents an N. Each row represents one possible outcome of the election, and the rows are listed in order of preference, with the voter's most preferred outcome at the top and least preferred outcome at the bottom.

10.14. The binary preference matrix corresponding Pete's preferences is symmetric, but those corresponding to Dave's and Mike's preferences are not.

10.17. The binary preference matrix corresponding to the preferences in Question 10.8 is not symmetric. Thus, the preferences in Question 10.8 are not separable.

10.18. The voter's preferences do not have to be separable. You may need to look at referendum elections with more than two proposals to find an example of this.

10.19. (a) The voter's preferences are not separable since the binary preference matrix is not symmetric.

(b) Each proposal by itself is separable with respect to the voter. Also, the first and third proposals together are separable with respect to the voter, as are the second and third proposals together. No other collection of proposals is separable with respect to the voter.

(c) For a voter whose preferences correspond to the binary preference matrix in this question, the first two proposals individually are each separable with respect to the voter, but together they are not separable. Thus, the statement is false.

10.21. The following collections of proposals would also have to be separable with respect to the voter: $\{\mathrm{A}, \mathrm{C}\}$, $\{\mathrm{B}, \mathrm{C}\}$, $\{\mathrm{C}, \mathrm{D}\}$, and $\{\mathrm{C}\}$.

10.29. The outcome under the sequential system would be N/Y, which is preferred by two of the three voters over the outcome under simultaneous voting.

10.35. (a) Dave could ensure a better result by voting N/Y instead of Y/N. The outcome would then be N/Y, which he prefers to N/N.

(b) Mike would not have an incentive to change his vote, since the result after Dave's change is N/Y—Mike's most preferred outcome. The only outcome Pete prefers to N/Y is N/N, and his vote of N/N already gives him the best chance of obtaining this outcome.

(c) Based on the answers to parts (a) and (b), it seems likely that iterative voting would stabilize on an outcome of N/Y.