Chapter 11

Proportional (Mis)representation

Focus Questions

In this chapter, we'll explore the following questions:

- What method is currently used to apportion the seats in the U.S. House of Representatives? What other methods have been used in the past?
- What is the quota rule? Which apportionment methods satisfy the quota rule, and which methods violate it?
- What are some examples of apportionment paradoxes? Is every apportionment method capable of producing paradoxes?
- Which apportionment method is the best? Is there any one method that is perfect or at least better than the others?

Warmup 11.1. Round off the following fifteen numbers (i.e., turn them into whole numbers) so that the sum of the rounded numbers equals the sum of the unrounded numbers (which is exactly 105):

$$6.408,\ \ 1.594,\ \ 2.226, 1.987, 8.622, 12.814,\ \ 3.826, 4.965,$$
$$9.175, 10.651, 11.693, 1.864, 6.716,\ \ 2.301, 20.158.$$

What rounding method did you use? Describe in detail how you decided which numbers to round up to the next whole number and which to round down to the previous whole number.

So what did you do with the fifteen numbers in Warmup 11.1? The most obvious thing to do would be to round them *conventionally*—that is, to round up if the number's decimal part is bigger than or equal to .50, and round down otherwise. However, there is a problem with this method: the sum of the rounded numbers equals 106, one more than the sum of the unrounded numbers. So you would have to reduce one of the rounded numbers by one, but which one? Would you reduce the one you rounded up whose decimal part was the smallest? Or maybe the one whose decimal part was the smallest *percentage* of the entire number? (For example, the decimal part of 1.594 is $\frac{.594}{1.594} = 37.26\%$ of the entire number, while the decimal part of 8.622, although larger in size, is only $\frac{.622}{8.622} = 7.21\%$ of the entire number.) Unless you were feeling particularly creative, you probably used one of these two rounding methods—most likely the first.

Question 11.2. Repeat Warmup 11.1, but use one or both of the two rounding methods described in the previous paragraph that you didn't use before.

Which rounding method do you think is better? And why does it matter in the first place? Well, the answers to these questions are more complicated than you might expect, and that is precisely why this chapter and the subject of *apportionment* exist. Apportionment basically deals with the problem of rounding off collections of numbers so that the sum of the rounded numbers equals the sum of the unrounded numbers.[1]

As dry as that might seem, apportionment has a substantial application in the political world, and is in fact at the center of a fascinating piece of American political history. This is because the most important type of apportionment problem is that of allocating seats in a legislative body to a collection of states or districts—and doing so in a way that yields a representation that is proportional to the populations of the states or districts. This problem first arose when our Founding Fathers were trying to decide how to distribute the seats in the U.S. House of Representatives to the original states.

The U.S. House of Representatives

It all started way back on May 25, 1787, when delegates from twelve of the original thirteen colonies met in Philadelphia for the Constitutional Convention. The most intense debate at the convention concerned the organization of the new country's legislature. The larger states wanted representation to be proportional to the states' populations. But, of course, the smaller states wanted all states to have equal representation. The delegates at the convention, in a masterful stroke of ingenuity, came up with a solution that

[1]According to Webster's Dictionary, the word *apportion* means "to divide and share out according to a plan; to make a proportionate division or distribution of."

satisfied everyone—both equal and proportional representation, in the Senate and House of Representatives, respectively. This is described in Article 1, Section 1 of the Constitution.

Article 1, Section 2 describes how the House of Representatives should be formed. Clearly, the intent of what is stated there is that the representatives should be allocated to the states based on their populations, but it is never written anywhere in the Constitution exactly how this allocation should be done. The writers of the Constitution surely didn't exclude this important detail without reason. They probably thought it was something that could be decided fairly easily at a more appropriate time, like perhaps a few years later when the initial apportionment of the seats in the House would take place. But, as it turns out, this omission from the Constitution led to much debate, anger, and ongoing frustration among the members of Congress, as well as much research and study by scholars concerning the mathematical problem of apportionment.

In defense of the writers of the Constitution, it's likely that they honestly had no idea just how serious the problems are that result from the noble goal of proportional representation. Apportionment had never really been studied in detail before the U.S. House was formed, and on the surface, the apportionment problem doesn't look too bad. But what happens when we look a bit deeper?

Let's start with the very first apportionment problem Congress ever faced. It occurred in 1794, following and based upon the population figures from the first national census, which took place in 1790. By that time there were fifteen states, and the census of 1790 counted the figures shown in Table 11.1.[2]

Congress needed to allocate exactly 105 seats in the House of Representatives to these fifteen states. After some debate, an apportionment bill authored by Alexander Hamilton was passed by Congress and forwarded to President George Washington.

Hamilton's Apportionment Method

Any apportionment method starts with the idea of a **standard quota**, which is just the exact number of seats a state would be entitled to based on its population. For example, since $\frac{59{,}096}{3{,}893{,}874} = 1.518\%$ of the entire U.S. population lived in Delaware in 1790, Delaware was entitled to exactly $.01518 \times 105 = 1.594$ seats in the first apportioned U.S. House of Representatives. We call 1.594 the standard quota for Delaware, since it is the

[2]The population figures actually used by Congress for the 1794 apportionment were slightly different from the figures shown in Table 11.1, though, since the figures actually used did not fully include the number of slaves and native Americans who lived in the U.S. in 1790.

State	Population
Connecticut	237,655
Delaware	59,096
Georgia	82,548
Kentucky	73,677
Maryland	319,728
Massachusetts	475,199
New Hampshire	141,899
New Jersey	184,139
New York	340,241
North Carolina	395,005
Pennsylvania	433,611
Rhode Island	69,112
South Carolina	249,073
Vermont	85,341
Virginia	747,550
TOTAL	3,893,874

Table 11.1. Population totals by state, 1790

exact number of seats Delaware would have been entitled to if fractional seats had been possible.

But herein lies the very essence of the apportionment problem: fractional seats are not possible! So Delaware, while entitled to its standard quota of 1.594 seats, had to be given a whole number of seats. Assuming this number is either 1 or 2, which number should it have been given? It would seem logical for Delaware to receive two seats, since this is the whole number that 1.594 rounds to conventionally. But conventional rounding, if applied similarly to the standard quotas for the other fourteen states, might not allocate a total of exactly 105 seats, just like conventionally rounding the numbers in Warmup 11.1 does not yield rounded numbers whose sum is 105. In fact, the standard quotas for the fifteen states shown in Table 11.1 are exactly the 15 numbers from Warmup 11.1. So if we tried to do the 1794 apportionment of the seats in the House by rounding the standard quotas for the states conventionally, we would give away 106 seats, one too many. What should we do then instead of conventional rounding so that we give away exactly 105 seats? Well, here's what Alexander Hamilton said to do:

Hamilton Step 1: Find the standard quota for each state.

Hamilton Step 2: Give each state a number of seats equal to its standard quota rounded down.

Hamilton Step 3: See how many seats are left to be allocated (called *surplus seats*), and give those seats one at a time to the states whose standard quotas have the largest decimal parts.

Question 11.3.* Use Hamilton's method to apportion 105 seats to the fifteen states with population figures shown in Table 11.1. Which states were winners under Hamilton's method, and which were losers?

At this point, our little U.S. history lesson takes its first unexpected turn. When President George Washington received a bill from Congress approving Hamilton's method for the 1794 apportionment of the seats in the House, he rejected it and sent it back stamped with the first presidential veto in U.S. history!

There has been considerable speculation as to why Washington vetoed the bill. Some conspiracy theorists claim he did so because his home state, Virginia, was a loser in the method, receiving 20 seats despite a standard quota of 20.158. Others claim he did so at the urging of Thomas Jefferson, who was Washington's Secretary of State, a personal friend, and a fellow Virginian. And indeed, the apportionment method that was eventually adopted was authored by Jefferson himself, and gave Virginia 21 seats. We will choose to believe, however, that Washington's veto was not motivated by self-interest, as Hamilton's method has some serious problems that reveal themselves upon closer inspection.

For one thing, decimal parts of standard quotas aren't always directly comparable. For example, in Question 11.3, Hamilton's method gives a surplus seat to Maryland, whose standard quota has a decimal part of .622, instead of Delaware, whose standard quota has a decimal part of .594—even though Delaware's decimal part is a much larger percentage of its entire standard quota than Maryland's.

Question 11.4.* For the standard quotas for Delaware and Maryland that you determined in Question 11.3, calculate the percentage that each state's decimal part is of its entire standard quota.

Question 11.5.

(a) For the standard quotas for each of the fifteen states in Question 11.3, which state's decimal part makes up the largest percentage of its entire standard quota? Which makes up the smallest percentage?

(b) In light of your answer to part (a), which state do you think was treated best in the apportionment from Question 11.3? Which do you think was treated worst?

In both of Questions 11.3 and 11.5, we looked at the decimal part of each state's standard quota to decide how the state was treated by Hamilton's apportionment method. But, as you might imagine, we could use other criteria as well. For instance, we could consider, for each state, the average number of residents represented by each of the state's representatives.

Question 11.6.* In the apportionment from Question 11.3, find the average number of residents represented by each of Delaware's representatives. Then repeat this calculation for each of the other states.

Question 11.7. According to your calculations from Question 11.6, which state was treated best in the apportionment from Question 11.3. Which state was treated worst?

So what should we do about poor Delaware? It was the first state, after all. But, it's really hard to justify giving Delaware two seats instead of one, as the next question demonstrates.

Question 11.8. Repeat Question 11.6, but this time assume Delaware had two representatives instead of just one. How does Delaware compare to the other states now?

And what about Rhode Island? Does it really deserve to be treated best? Let's consider what would happen to the final figures in Question 11.6 if we took a seat from Rhode Island. Of course, to keep the total number of representatives at 105, we would have to give that seat to some other state. Since Virginia's population is so much larger than any of the other states, its final figure in Question 11.6 would be the least affected if we gave it the extra seat. So let's do this and see what happens.

Question 11.9.* Repeat Question 11.6, but this time assume Rhode Island had just 1 representative instead of 2, and Virginia had 21 representatives instead of just 20. (Note: The calculations from Question 11.6 will be different only for Rhode Island and Virginia.) How do Rhode Island and Virginia compare to the other states now? Explain.

So do you think we made the system better by taking a seat from Rhode Island and giving it to Virginia? Well, George Washington and Thomas Jefferson would say yes, and not just because they were Virginians.

Jefferson's Apportionment Method

When President Washington vetoed the bill approving Hamilton's method for the 1794 apportionment, Congress did not have enough votes to override it. So instead they passed a bill approving an apportionment method proposed by Thomas Jefferson.

Jefferson's method is a **divisor method**. To see what this means, consider again the calculations used to find standard quotas. For example, with the population figures shown in Table 11.1 and a total of 105 seats to be apportioned, the standard quota for Connecticut is

$$\frac{237{,}655}{3{,}893{,}874} \times 105 = 6.408,$$

and the standard quota for Delaware is

$$\frac{59{,}096}{3{,}893{,}874} \times 105 = 1.594.$$

Note that the only input numbers different in these two calculations are the populations of the states (the top numbers in the fractions on the left). In fact, the standard quota for each state could be calculated the same way—by dividing the state's population by 3,893,874 (the total population of all the states), and then multiplying the result by 105 (the total number of seats). The only quantity that would vary from one calculation to the next would be the population of the state itself. For this reason, standard quota calculations are often expressed in a slightly different, but mathematically equivalent, form. For example, Connecticut's standard quota calculation can be expressed as

$$\frac{237{,}655}{\frac{3{,}893{,}874}{105}} = 6.408,$$

and Delaware's as

$$\frac{59{,}096}{\frac{3{,}893{,}874}{105}} = 1.594.$$

Expressing these calculations in this way makes them a bit harder to follow. But this alternative form is useful in that it allows us to more easily identify the role of the divisor $\frac{3{,}893{,}874}{105}$ (which is 37,084.51) in each state's standard quota calculation. This value is called the **standard divisor**, and is completely determined by the total population of the system and the total number of seats to be apportioned, neither of which would change if the calculation were done for a different state. So, once the standard divisor for a system has been found, the standard quota for each state can be found by simply dividing the state's population by the standard divisor.

Question 11.10. In the apportionment from Question 11.3, use the standard divisor find the standard quotas for all of the states.

The reason Jefferson's method is called a divisor method is because it works by modifying the standard divisor until a modified divisor results in modified quotas that give away exactly the correct number of seats when they are all rounded according to the same common convention. The specific details of Jefferson's method are as follows:

Jefferson Step 1: Find the standard quota for each state.

Jefferson Step 2: Round each standard quota down, and check to see if the sum of the rounded standard quotas equals the total number of seats to be apportioned. If so, the method is complete. Otherwise, continue with Step 3.

Jefferson Step 3: Choose a modified divisor that is different from the standard divisor, and use this modified divisor to calculate modified quotas for each state (by dividing each state's population by the modified divisor).

Jefferson Step 4: Round each modified quota down, and check to see if the sum of the rounded modified quotas equals the total number of seats to be apportioned. If so, the method is complete. Otherwise, repeat Steps 3 and 4 (with a different modified divisor) until the method is complete.

Note that steps 3 and 4 in Jefferson's method include some inherent trial and error, a fact that has the potential to make Jefferson's method significantly more time-consuming than Hamilton's. Jefferson's method, however, does have one clear advantage—it rounds all of the quotas according to the same common convention. This is unlike Hamilton's method, in which a particular decimal part might in some instances be rounded up, while in other instances be rounded down. In fact, the consistency of the rounding convention employed by Jefferson's method is likely what caused Washington and Jefferson to believe it was fairer than Hamilton's.

Question 11.11.*

(a) Use Jefferson's method to apportion 105 seats to the fifteen states with population figures shown in Table 11.1. Write down the apportionment that results from each modified divisor you try, including those that fail to give away exactly 105 seats.

(b) In your apportionment from part (a), which states were treated best, and which were treated worst?

Despite the fact that Hamilton's method is easier to use than Jefferson's, it was Jefferson's method that was used to apportion the seats in the House in 1794 (although, as we noted earlier, the population figures used in 1794 were slightly different from those shown in Table 11.1). If you completed Question 11.11 correctly, the apportionment you found using Jefferson's method should differ from the apportionment you found using Hamilton's method (in Question 11.3), but only in the fact that Jefferson's method should have caused Rhode Island to lose a seat to Virginia. So we see that, although it is in fact possible for Hamilton's and Jefferson's methods to yield identical apportionments for the same system, they don't have to. And actually, with the slightly different population figures that were actually used in 1794, Hamilton's method gave both Delaware and Rhode Island a second seat, and it was Delaware that lost its second seat to Virginia under Jefferson's method, leaving Rhode Island still with two.

What's really important to note about this is that in both the final apportionment from Question 11.11 and the actual apportionment from 1794, a small state lost a seat to a large state when compared to what the results would have been for the same system under Hamilton's method. This phenomenon actually occurs quite regularly under Jefferson's method, and the next question and subsequent discussion reveals why.

Question 11.12. Explain why in step 2 of Jefferson's method, too few seats will always be given away (except in the essentially impossible event that the original standard quotas are all exact whole numbers).

Thus, after step 2 of Jefferson's method, more seats will always need to be given away. As a result, in Jefferson's method, the standard divisor will always need to be modified lower, since lowering the divisor will increase the quotas (because we will be dividing by a smaller number). However, modifying a divisor lower will cause larger quotas to increase more quickly than smaller ones. For example, notice how much bigger the modified quotas for Rhode Island and Virginia are in Question 11.11 than their respective standard quotas. (Virginia's modified quota should have increased from its standard quota much more than Rhode Island's.)

So, in general, larger quotas have a greater chance than smaller quotas of increasing over the next whole number when the divisor is lowered. Thus, states with larger quotas (and correspondingly larger populations) have a better chance than states with smaller quotas of receiving additional seats under Jefferson's method. In fact, if one state's quota is significantly larger than those of other states (as Virginia's was in 1790), the larger quota might increase all the way over the next *two* whole numbers before all of the additional seats have been given away. The next question illustrates this phenomenon.

Question 11.13. The census of 1820 recorded populations of 1,368,775 for the state of New York, and 8,969,878 for the entire U.S. Based upon the populations of New York and the other states recorded in this census, 213 seats in the House were to be apportioned in 1822.

(a) Using the 1820 census data, calculate the standard divisor and New York's standard quota.

(b) In the apportionment of 1822, Jefferson's method was used with a modified divisor of 39,900. Using this modified divisor, find New York's modified quota and the final number of seats the state was given. Do you think it was fair that New York received this number of seats? Why or why not?

The apportionment of 1822 revealed a serious flaw in Jefferson's method—as we saw in part (b) of Question 11.13. Unfortunately, though, Congress did nothing in response. Perhaps it was believed that the problem was a fluke, an anomaly that would not occur again or at least not occur often enough to be a real cause for concern. But then the same thing occurred in the very next apportionment, in 1832 (based on data from the census of 1830), when Jefferson's method gave New York 40 seats in the House even though its standard quota was only 38.59.

At this point, the problem had to be addressed. Daniel Webster, among many others, was outraged. Webster, an accomplished orator, in one of his more famous speeches, argued vehemently before Congress that apportioning 40 seats to New York was not only troublesome, but unconstitutional.

Two alternative apportionment methods were subsequently presented to Congress. One was proposed by John Quincy Adams and is identical to Jefferson's method, except that in steps 2 and 4 of Adams' method, quotas are rounded up rather than down.

Question 11.14.*

(a) Use Adams' method to apportion 105 seats to the fifteen states with population figures shown in Table 11.1. Write down the apportionment that results from each modified divisor you try, including those that fail to give away exactly 105 seats.

(b) In your apportionment from part (a), which states were treated best, and which were treated worst?

Question 11.15. In step 3 of Adams' method, how will the standard divisor always need to be modified: higher or lower? Give a convincing argument to justify your answer.

Since Adams' method is just the mirror image of Jefferson's, it is obviously no better. Just as Jefferson's method works consistently and unfairly in favor of larger states, Adams' works consistently and unfairly in favor of smaller states. So Adams' method was never actually used to apportion the seats in the U.S. House. But we will give Adams the benefit of the doubt and suppose that maybe he just carried to an extreme level our Founding Fathers' desire to provide protection for the smaller states in the Union.

The second apportionment method presented to Congress as an alternative to Jefferson's method was proposed by Daniel Webster himself. We'll investigate Webster's method in the next section, but before we do so, let's first formalize one of our observations about Jefferson's method.

As we noted, New York was given 40 seats in the 1832 apportionment, even though its standard quota was only 38.59. Using current terminology, we would say that this violates the **quota rule**. The quota rule states that, in an apportionment, each state should be given a number of seats equal to its original standard quota, rounded either up or down. An apportionment method for which the quota rule always holds is said to *satisfy quota*. Jefferson's method *violates quota* because it is possible for a state to receive more seats than its standard quota rounded up. (Adams' method also violates quota, as it is the mirror image of Jefferson's.) Moreover, it is actually very common for Jefferson's method to violate quota. As it turns out, if Jefferson's method had continued to be used, every apportionment of the House since 1852 would have violated quota.[3]

However, 1832 was the end of Jefferson's method, which brings us back to Daniel Webster.

Webster's Apportionment Method

The apportionment method proposed by Daniel Webster in 1832 was, like Adams' method, a divisor method that differed from Jefferson's only in the rule used for rounding. In particular, Webster's method used conventional rounding (instead of always rounding down) in steps 2 and 4.

Question 11.16.*

(a) Use Webster's method to apportion 105 seats to the fifteen states with population figures shown in Table 11.1. Write down the apportionment that results from each modified divisor you try, including those that fail to give away exactly 105 seats.

(b) In your apportionment from part (a), which states were treated best, and which were treated worst?

Question 11.17. Clearly explain why in step 3 of Webster's method, the standard divisor could need to be modified either higher or lower.

Webster's method was used for the apportionment of 1842, and it is regarded by many modern-day experts as the best of all apportionment methods. One reason for this high regard is the fact that Webster's method is completely neutral in how it treats larger states in comparison to smaller states. In more precise terms, Webster's method slightly favors smaller states when conventional rounding of the standard quotas gives away too many seats, and larger states when conventional rounding of the standard quotas gives away too few seats. But since conventional rounding of standard quotas is equally likely to give away too many seats as too few, the method is ultimately neutral.

It has been shown that Webster's method (and, indeed, any divisor method) can violate quota, but examples of such violations tend to be so contrived that they would occur only very rarely in real-life situations. In fact, if Webster's method had been used consistently from the first apportionment of the House in 1794 to the most recent reapportionment in 2012, it would still have yet to violate quota even once.

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[3]After the first apportionment of the House in 1794, the seats were then supposed to be reapportioned every decade on years ending in 2, using the population figures recorded in the national census from two years prior.

Even so, the very possibility of violating quota left Congress leery of Webster's method, especially after the fiasco of 1832. So, in 1850, Congressman Samuel Vinton proposed what he believed to be a brand new apportionment method. As it turned out, it was identical to Hamilton's method. In Vinton's defense, nobody else remembered it either. So Congress called the new method Vinton's method, and, since it can never violate quota, received it warmly. (To avoid confusion, we'll continue to call it Hamilton's method.)

Question 11.18. Clearly explain why Hamilton's method can never violate quota.

In 1852, Congress passed a law adopting Hamilton's method as the official method of apportionment for the seats in the House. However, since in practice Hamilton's and Webster's methods often yield identical apportionments, an unofficial compromise was also adopted that in 1852 and future years, Congress would have a total number of seats in the House for which Hamilton's and Webster's methods would yield identical apportionments. And since members of Congress do not typically like to legislate themselves out of business, we could better describe this compromise by saying that, in 1852 and future years, Congress would *increase* the total number of seats in the House to a number for which Hamilton's and Webster's methods would yield identical apportionments.

This agreement lasted only until 1872, when, in direct violation of the Constitution (which specifies that some prescribed method must be used to apportion the seats in the House) and illegally (violating the 1852 law designating Hamilton's method as the official method), Congress passed an apportionment bill that wasn't based on any method at all. The apportionment gods got their revenge in the presidential election of 1876, when Rutherford B. Hayes defeated Samuel Tilden thanks to Electoral College numbers resulting from the unconstitutional apportionment of 1872. If either Hamilton's or Webster's method had been used to apportion the seats in the House in 1872, Tilden would have easily won the election. Undoubtedly with great humility, Congress went back to Hamilton's method in 1882, which is where this story takes yet another unexpected turn.

Three Apportionment Paradoxes

As part of the procedure for apportioning the seats in the House of Representatives in 1882, the U.S. Census Bureau supplied Congress with a table showing the apportionments under Hamilton's method for all sizes of the House between 275 and 350 seats. This table revealed something truly bizarre.

With a House size of 299 seats, Alabama's standard quota was 7.646, Illinois' was 18.640, and Texas's was 9.640. A ranking of the 38 states at the time, starting with the one whose standard quota had the largest decimal part and ending with the one whose standard quota had the smallest, placed Alabama 20th. In addition, with a House size of 299 seats, there were exactly 20 surplus seats to be given away in Hamilton's method.

With a House size of 300 seats, the standard quotas for all of the states were naturally a bit larger. Alabama's increased to 7.671, Illinois' to 18.702, and Texas's to 9.672. This placed Illinois 20th in the new ranking of the states, with Texas next in line. And with a House size of 300 seats, there were 21 surplus seats to be given away instead of just 20.

Question 11.19. Based on the figures in the previous two paragraphs, how many seats would Hamilton's method have given to Alabama, Illinois, and Texas if the total size of the House in 1882 had been 299 seats? How many seats would each state have received if the total size of the House had been 300 seats? Does anything about these two potential apportionments strike you as being unusual or unfair? Explain.

The remarkably unfair phenomenon you observed in Question 11.19 is actually a relatively common occurrence under Hamilton's method when the states involved have highly varied populations. When it was observed in 1882, it finally validated George Washington's veto of Hamilton's method from no less than 88 years prior.

This unfair phenomenon also provides us with another example of a paradox, but in a completely different setting than what we've seen before. Just as Condorcet's voting paradox from Chapter 3 seems to defy logic in the voting world, in the apportionment world it contradicts common sense that increasing the number of seats in the House from 299 to 300 would cause Alabama to *lose* a seat. This apportionment paradox (when increasing the number of seats in an apportioned system, in and of itself, causes a state to lose a seat) is commonly referred to as the **Alabama paradox**.

So how did Congress resolve the paradox in 1882? They opted to go with a House size of 325 seats, a value for which the paradox did not present itself. Then they crossed their fingers and hoped the paradox would never be heard from again. And then 1902 happened.

Question 11.20. As part of 1902 apportionment, the U.S. Census Bureau supplied Congress with a table showing the apportionments under Hamilton's method for all sizes of the House between 350 and 400 seats. When the number of seats in the House was 357, 382, 386, 389, and 390, Maine would be given three seats, but for all other House sizes Maine would be given four seats. Also, when the number of seats in the House was 357, Colorado would be given two seats, but for all other House sizes Colorado would be given three seats. Based only on these figures for Maine and Colorado, would the Alabama paradox have occurred in 1902 for at least one House size between 350 and 400 seats (assuming Hamilton's method was used)? If so, for which House sizes and which states?

Naturally then, a bill was presented in Congress to apportion the seats in the House in 1902 using Hamilton's method with a House size of 357 seats. (We'll pause for a moment to let that sink in.) As you might suspect, the resulting debate got a little heated. In the end, the bill was defeated, and Hamilton's method was scrapped once and for all. The final 1902 apportionment was done using Webster's method with a House size of 386 seats.

The would-be occurrence of the Alabama paradox in 1902 was the final death blow for Hamilton's method, and it has never again been used to apportion the seats in the House. However, scholars have continued to study Hamilton's method, and two additional paradoxes that can occur under the method have been discovered.

One, called the **population paradox**, was also discovered around the time of the 1902 apportionment, when it was observed that Hamilton's method would have caused Virginia to lose a seat to Maine, even though the population of Virginia had grown by a larger percentage over the previous decade than that of Maine.

Question 11.21.* According to the U.S. Census Bureau, the population of Nevada grew from 1,998,257 in 2000 to 2,700,551 in 2010, and the population of Illinois grew from 12,419,293 in 2000 to 12,830,632 in 2010. In the 2012 apportionment, which was based on the Census Bureau's population figures from 2010, Nevada gained a seat in the House while Illinois lost a seat. Based on these figures for Nevada and Illinois, can you conclude that the population paradox occurred in 2012? Why or why not?

Question 11.22. According to the U.S. Census Bureau, the population of Nevada grew from 1,201,598 in 1990 to 1,998,257 in 2000, and the population of Illinois grew from 11,435,813 in 1990 to 12,419,293 in 2000. In the 2002 apportionment, which was based on the Census Bureau's population figures from 2000, Nevada gained a seat in the House while Illinois lost a seat. Based on these figures for Nevada and Illinois, can you conclude that the population paradox occurred in 2002? Why or why not? And why is this question more interesting than Question 11.21?

Another paradox that can occur under Hamilton's method was discovered in 1907, when Oklahoma joined the Union as the 46th state. Because it was not yet time for the next reapportionment, Congress decided to simply increase the size of the House and give Oklahoma a number of seats proportional to its population. This decision resulted in 5 seats being added to the House, increasing its total size from 386 to 391. However, it was then noted that if Hamilton's method had been used to apportion 386 seats to the 45 states in 1902, and then again to apportion 391 seats to the 46 states in 1907, New York would have lost a seat to Maine! In other words, adding a new state and its fair share of seats would have caused changes (both positive and negative) in the number of seats given to other existing states. Incidents such as this are examples of what is now commonly called the **new-states paradox**.

The discovery of the population and new-states paradoxes supported the decision by Congress in 1902 to stop using Hamilton's method. And these paradoxes, as well as the Alabama paradox, will never bother Congress again, as it turns out they are impossible under divisor methods such as Jefferson's, Adams', and Webster's. (See Question 11.32.)

Hill's Apportionment Method

Although, at this point, our history lesson has only brought us as far as 1907, we're actually nearing the end of the apportionment story. We have but a single new method left to discuss—the one currently used to apportion the seats in the U.S. House of Representatives.

As we mentioned in the previous section, Webster's method was used for the 1902 apportionment. For the 1912 apportionment, Walter Willcox, the first American to investigate apportionment from a theoretical perspective, argued successfully to Congress that Webster's method should again be used. Around this same time, Joseph Hill, Chief Statistician of the U.S. Census Bureau, proposed a new method—with a strong endorsement from famed mathematician Edward Huntington.

Hill's method is yet another divisor method, and it is almost exactly identical to Webster's. The one lone difference between Webster's and Hill's methods is in the location of the cutoffs used for rounding quotas. Under Webster's method, a quota is rounded up if its decimal part is bigger than or equal to .50, and down otherwise. We can view this .50 cutoff as the decimal part of the average, or *arithmetic mean*, of the two whole numbers closest to the quota. For example, under Webster's method the quota 5.482 would be rounded down because its decimal part is smaller than the decimal part of 5.50, which is the arithmetic mean of 5 and 6.

Hill's method, instead of using an arithmetic mean to determine the cutoff for rounding a quota up or down, uses the *geometric mean* of the two whole numbers closest to the quota. The geometric mean of any two whole numbers x and y is simply $\sqrt{x \cdot y}$. So, for Hill's method, a quota is rounded up if its decimal part is bigger than or equal to the decimal part of the geometric mean of the two closest whole numbers, and down otherwise. For example, for the quota 5.482, the geometric mean of the two closest whole numbers is $\sqrt{5 \cdot 6} = 5.477$. So, because the decimal part of 5.482 is bigger than the decimal part of 5.477, under Hill's method the quota 5.482 would be rounded up to 6. But (and notably!), since $\sqrt{15 \cdot 16} = 15.492$, under Hill's method the quota 15.482 would be rounded down to 15.

Question 11.23.*

(a) Use Hill's method to apportion 105 seats to the fifteen states with population figures shown in Table 11.1. Write down the apportionment that results from each modified divisor you try, including those that fail to give away exactly 105 seats.

(b) In your apportionment from part (a), which states were treated best, and which were treated worst?

(c) How does the apportionment you found using Hill's method in part (a) compare to those you found for this same system in previous questions using Hamilton's, Jefferson's, Adams', and Webster's methods?

Question 11.24. Is Hill's method biased in favor of larger states, smaller states, or neither? (Hint: See the last two sentences before Question 11.23.)

Both Webster's and Hill's methods were considered for the 1922 apportionment, and the two methods produced significantly different outcomes. By this time the number of seats in the House had been fixed by law, and so as a result of the discrepancy between Webster's and Hill's methods, no apportionment bill was passed. Consequently, the 1912 seat totals were held over without any reapportionment whatsoever. (This was, of course, another direct violation of the Constitution.)

In preparation for the 1932 apportionment, a committee of members from the National Academy of Sciences was formed to study Webster's and Hill's methods. The committee endorsed Hill's method, a powerful victory in the contest between the two. But then, in a remarkable twist of fate, Webster's and Hill's methods produced identical apportionments using the 1930 census data. So proponents of either method could claim that theirs was the one used in 1932.

For the 1942 apportionment, Webster's and Hill's methods came very close to again producing identical apportionments. The only difference between the two was that Hill's method gave a single extra seat to Arkansas at Michigan's expense. At the time, Michigan tended to elect Republican legislators, while Arkansas tended to elect Democrats. The vote on the resulting apportionment bill split exactly along party lines, with the Democrats supporting Hill's method and the Republicans supporting Webster's. Because the Democrats had the majority, it was Hill's method that passed through Congress. President Franklin D. Roosevelt, also a Democrat, signed the method into "permanent" law, and it has been used for every reapportionment of the House since.

Another Impossibility Theorem

Both Webster's and Hill's methods are held in high regard by scholars who study apportionment. In fact, most modern-day experts are proponents of one of the two methods, and with good reason—both are relatively neutral in how they balance power between small and large states, and both are incapable of producing the three apportionment paradoxes we saw earlier.

With that said, both Webster's and Hill's methods can violate quota, which begs the question: Wouldn't it be nice if we could find an apportionment method that was incapable of producing paradoxes *and* never violated quota? Well, in the 1970s, mathematicians Michel Balinski and Peyton Young set out to find such a system. The result of their search might have surprised us way back in Chapter 1, but not any longer.

Balinski and Young's Theorem. *It is impossible for an apportionment method to always satisfy quota and be incapable of producing paradoxes.*

The basic idea behind the proof of Balinski and Young's Theorem is actually quite simple. They first showed that the only apportionment methods that avoid the population paradox are divisor methods. But it was already known before Balinski and Young that every divisor method is capable of violating quota. So, in order to avoid the population paradox, you need a divisor method. And as soon as you have a divisor method, you run the risk of violating quota. Balinski and Young's Theorem follows easily from these two facts.

Question 11.25. In light of Balinski and Young's Theorem, which do you think is more important for an apportionment method: that it never violate quota or that it avoid the population paradox? (Remember, you can't have both!) Give a convincing argument to justify your answer.

Like Arrow's Theorem did in the world of voting, Balinski and Young's Theorem takes us full circle in the world of proportional representation; it shows us that, as was the case with voting, proportional representation is incapable of being free from controversy.

Even so, some apportionment methods are clearly better than others. For instance, Hamilton's method is certainly the easiest to use, and that is why, despite its drawbacks, it is still the method of choice in a number of countries around the world. On the surface, Webster's method seems to be the fairest from a mathematical perspective. But Hill's was the method endorsed by the National Academy of Sciences after it was carefully compared to Webster's in the late 1920s. It should also be noted that when Balinski and Young first presented a proof of their famed impossibility theorem in 1980, they went on to argue convincingly why they believed Webster's method was the best.

So, as with voting, we may never arrive at a definitive answer to the apportionment problem. Nevertheless, what we've learned in this chapter at least gives us the tools we need to analyze various apportionment methods and approach proportional representation from a reasoned and logical perspective.

Concluding Remarks

In this chapter, we've looked at five different methods for apportioning seats in a legislative body based on the populations of the states or districts to be represented. In each case, for the sake of space, the only example we considered was the initial apportionment of the seats in the U.S. House of Representatives. The apportionment methods we investigated can be applied in a great variety of situations, depending on whether we are interested in apportioning a fixed number of legislative seats to counties in a state, or board of supervisors seats to districts in a city, or even pieces of candy to children in a family. The methods we investigated work exactly the same way in any situation in which objects to be distributed cannot be divided into smaller pieces.

If you wish to study apportionment in more detail, you may find yourself considering examples in other books or online. Be advised that, in other sources, the five apportionment methods we discussed may be referred to by different names. For instance, Hamilton's method is sometimes called the method of largest remainders, Jefferson's the method of greatest divisors or d'Hondt's method (as it is known in Europe), Adams' the method of smallest divisors, Webster's the method of major fractions or the Webster-Willcox method, and Hill's the method of equal proportions or the Hill-Huntington method.

Questions for Further Study

Question 11.26. When the first national U.S. census was conducted in 1790, Maine was still considered part of Massachusetts. If Maine had been considered a separate state at the time, then the 105 seats in the initial 1794 apportionment of the House would have been distributed among sixteen states instead of fifteen. Assuming the population of Maine in 1790 was 96,643, use the population figures in Table 11.1 to recalculate the 1794 apportionment, viewing Maine as a separate state. Use at least two of the apportionment methods we looked at in this chapter, and write a summary comparing the resulting apportionments both to each other and to the apportionments you calculated in this chapter with only fifteen states. (Note: When you calculate a new apportionment, don't forget to adjust the population of Massachusetts by subtracting Maine's from it.)

Question 11.27. Remember the Marquis de Condorcet? Well, as it turns out, he proposed an apportionment method too. His method was a divisor method, and his convention for rounding was to round a number up if its decimal part was bigger than or equal to .40, and down otherwise. Was Condorcet's method biased in favor of larger states, smaller states, or neither, or is it impossible to determine this from this information? Explain.

Question 11.28. Investigate Lowndes' apportionment method, and write a complete summary of your findings. Include in your summary a description of how the method works, at least two small numerical demonstrations of what "relative fractional parts" means, who first proposed the method and when this occurred, how the method compares to the other apportionment methods we looked at in this chapter, and—if you can determine so—whether the method can violate quota or produce any of the three apportionment paradoxes.

Question 11.29. Investigate Dean's apportionment method (also known as the method of harmonic means), and write a complete summary of your findings. Include in your summary a description of how the method works, at least two small numerical demonstrations of what the *harmonic mean* of two numbers is, who first proposed the method and when this occurred, how the method compares to the other apportionment methods we looked at in this chapter, and if you can determine so, whether the method can violate quota or produce any of the three apportionment paradoxes.

Question 11.30. Investigate which countries around the world use Hamilton's method to allocate seats in their governmental legislative bodies. Then investigate which countries use Jefferson's method.

Question 11.31. Which of the apportionment methods from this chapter (including the ones from Questions 11.27–11.29, if you did those questions) do you think is best, and why? Give a convincing argument to justify your answer.

Question 11.32. Why can divisor methods not produce any of the three apportionment paradoxes we looked at in this chapter? Either form your own explanation, or do some research and write a complete summary of your findings.

Question 11.33. Write a short biography of Walter Willcox, including his most important contributions both inside and outside of apportionment.

Question 11.34. Write a short biography of Edward Huntington, including his most important contributions both inside and outside of apportionment.

Question 11.35. Write short biographies of Michel Balinski and Peyton Young, including their most important contributions both inside and outside of apportionment.

Question 11.36. Find a copy of Article 1, Section 2 of the U.S. Constitution, and write a summary of what it states regarding apportionment. Then critique this section of the Constitution by identifying any deficiencies present in its description of how the seats in the U.S. House of Representatives should be apportioned to the states.

Question 11.37. Find a copy of George Washington's veto message when he vetoed Hamilton's apportionment method, and write a summary of what he stated. Then critique Washington's veto message. Do you think Washington's veto was personally motivated, or do you think he really saw some deficiencies in Hamilton's method?

Question 11.38. Find a copy of Daniel Webster's speech to Congress in which he argued that apportioning 40 seats to New York in 1832 was unconstitutional. Write a summary and critique of Webster's speech, evaluating his argument in light of what you learned in this chapter. Do you think Webster presented his case in the best possible way, or could his argument have been stronger?

Question 11.39. Find out exactly how the seats in the U.S. House of Representatives were apportioned in 1872, and write a complete summary of your findings.

Question 11.40. Investigate the results of the apportionments of the seats in the U.S. House of Representatives in 2002 and 2012, and write a summary comparing the two. Which states gained seats and which lost seats between 2002 and 2012? Do you think your state was treated fairly in the 2012 apportionment? Based on its population, is your state currently underrepresented, overrepresented, or perfectly represented in the House?

Question 11.41. In 1991, a lawsuit, *Montana* v. *United States Department of Commerce*, was filed in U.S. Federal District Court. Find out the details of this lawsuit and the subsequent Supreme Court ruling. Write a summary of your findings, including the outcomes of both the original lawsuit and the Supreme Court ruling, and your own personal feelings about what the outcomes should have been.

Question 11.42. Find an article in a popular media source that expresses a positive view of the method currently used to apportion the seats in the U.S. House of Representatives. Write a summary and critique of the article based on what you learned in this chapter.

Question 11.43. Find an article in a popular media source that expresses a negative view or questions the constitutionality of the method currently used to apportion the seats in the U.S. House of Representatives. Write a summary and critique of the article based on what you learned in this chapter.

Question 11.44. If Hamilton's method had been used for the 2012 apportionment of the House, would this have made a difference in the outcome of the 2016 U.S. presidential election? Completely explain your answer.

Question 11.45. If Jefferson's method had been used for the 2012 apportionment of the House, would this have made a difference in the outcome of the 2016 U.S. presidential election? Completely explain your answer.

Question 11.46. If Hamilton's method had been used for the 1992 apportionment of the House, would this have made a difference in the outcome of the 2000 U.S. presidential election? Completely explain your answer.

Question 11.47. If Jefferson's method had been used for the 1992 apportionment of the House, would this have made a difference in the outcome of the 2000 U.S. presidential election? Completely explain your answer.

Question 11.48. Create a spreadsheet in Microsoft Excel or a similar product to implement one or more of the apportionment methods we looked at in this chapter. (The only special Excel functions you should need are `ROUNDDOWN` for Hamilton's and Jefferson's methods, `ROUNDUP` for Adams', `ROUND` for Webster's, and `SQRT` for Hill's.) Then use your spreadsheet to apportion the U.S. House for some year in which seats were apportioned.

Question 11.49. In 1991, a lawsuit, *Commonwealth of Massachusetts* v. *Mosbacher*, was filed in Massachusetts District Court. Find out the details of this lawsuit, and write a summary of your findings. Include in your summary the outcome of the lawsuit, and your own personal feelings about what the outcome should have been.

Question 11.50. Find a copy of the article "Outcomes of Presidential Elections and the House Size," by Michael G. Neubauer and Joel Zeitlin, in the journal *PS: Political Science & Politics*. Write a summary of what the article states regarding the relationship between the apportionment of the U.S. House of Representatives and U.S. presidential elections.

Answers to Starred Questions

11.3. There should be a total of 9 surplus seats. In the correct apportionment, Connecticut should be given 6 seats, Delaware 1 seat, Maryland 9 seats, and Virginia 20 seats.

11.4. See the first paragraph after Warmup 11.1 for these percentages.

11.6. Delaware has one representative and a population of 59,096. Thus each of Delaware's representatives represents on average $\frac{59{,}096}{1} = 59{,}096$ residents. The calculations for the other states can be done similarly (and are perhaps more interesting).

11.9. With the revised numbers, Virginia is the fourth-best treated state, and Rhode Island is the worst (even worse than Delaware!).

11.11. A modified divisor of 35,000 produces the correct apportionment. (There are other modified divisors that produce the correct apportionment as well. Typically with divisor methods, there are multiple divisors that all give away exactly the correct numbers of seats.) The correct apportionment here should be identical to the one from Question 11.3, except that Rhode Island should have lost a seat to Virginia.

11.14. A modified divisor of 39,600 produces the correct apportionment. In comparison of the correct apportionment here and the one from Question 11.3, four states should have gained a seat and four should have lost a seat.

11.16. A modified divisor of 37,616 produces the correct apportionment. The correct apportionment here should be identical to the one from Question 11.3, except that Maryland should have lost a seat to Delaware.

11.21. The population of Nevada grew by $\frac{2{,}700{,}551-1{,}998{,}257}{1{,}998{,}257} = 35.1\%$ between 2000 and 2010, but during this same time period the population of Illinois grew by only $\frac{12{,}830{,}632-12{,}419{,}293}{12{,}419{,}293} = 3.3\%$. Thus we cannot conclude that the population paradox occurred.

11.23. A modified divisor of 37,670 produces the correct apportionment. The correct apportionment here should be identical to the one from Question 11.3, except that North Carolina should have lost a seat to Delaware.