1.2 GAUSSIAN ELIMINATION AND MATRICES

The problem is to calculate, if possible, a common solution for a system of m linear algebraic equations in n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m,
\end{aligned}
$$

where the x_i 's are the unknowns and the a_{ij} 's and the b_i 's are known constants. The a_{ij} 's are called the ***coefficients*** of the system, and the set of b_i 's is referred to as the ***right-hand side*** of the system. For any such system, there are exactly three possibilities for the set of solutions.

Three Possibilities

- **UNIQUE SOLUTION:** There is one and only one set of values for the x_i 's that satisfies all equations simultaneously.
- **NO SOLUTION:** There is no set of values for the x_i 's that satisfies all equations simultaneously—the solution set is empty.
- **INFINITELY MANY SOLUTIONS:** There are infinitely many different sets of values for the x_i 's that satisfy all equations simultaneously. It is not difficult to prove that if a system has more than one solution, then it has infinitely many solutions. For example, it is impossible for a system to have exactly two different solutions.

Part of the job in dealing with a linear system is to decide which one of these three possibilities is true. The other part of the task is to compute the solution if it is unique or to describe the set of all solutions if there are many solutions. Gaussian elimination is a tool that can be used to accomplish all of these goals.

Gaussian elimination is a methodical process of systematically transforming one system into another simpler, but equivalent, system (two systems are called ***equivalent*** if they possess equal solution sets) by successively eliminating unknowns and eventually arriving at a system that is easily solvable. The elimination process relies on three simple operations by which to transform one system to another equivalent system. To describe these operations, let E_k denote the k^{th} equation

$$E_k: \quad a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k$$

and write the system as

$$\mathcal{S} = \left\{ \begin{matrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{matrix} \right\}.$$

For a linear system $\mathcal{S}$, each of the following three ***elementary operations*** results in an equivalent system $\mathcal{S}'$.

(1) Interchange the i^{th} and j^{th} equations. That is, if

$$\mathcal{S} = \left\{ \begin{matrix} E_1 \\ \vdots \\ E_i \\ \vdots \\ E_j \\ \vdots \\ E_m \end{matrix} \right\}, \quad \text{then} \quad \mathcal{S}' = \left\{ \begin{matrix} E_1 \\ \vdots \\ E_j \\ \vdots \\ E_i \\ \vdots \\ E_m \end{matrix} \right\}. \tag{1.2.1}$$

(2) Replace the i^{th} equation by a nonzero multiple of itself. That is,

$$\mathcal{S}' = \left\{ \begin{matrix} E_1 \\ \vdots \\ \alpha E_i \\ \vdots \\ E_m \end{matrix} \right\}, \quad \text{where} \quad \alpha \neq 0. \tag{1.2.2}$$

(3) Replace the j^{th} equation by a combination of itself plus a multiple of the i^{th} equation. That is,

$$\mathcal{S}' = \left\{ \begin{matrix} E_1 \\ \vdots \\ E_i \\ \vdots \\ E_j + \alpha E_i \\ \vdots \\ E_m \end{matrix} \right\}. \tag{1.2.3}$$

Providing explanations for why each of these operations cannot change the solution set is left as an exercise.

The most common problem encountered in practice is the one in which there are n equations as well as n unknowns—called a ***square system***—for which there is a unique solution. Since Gaussian elimination is straightforward for this case, we begin here and later discuss the other possibilities. What follows is a detailed description of Gaussian elimination as applied to the following simple (but typical) square system:

$$\begin{aligned} 2x + y + z &= 1, \\ 6x + 2y + z &= -1, \\ -2x + 2y + z &= 7. \end{aligned} \tag{1.2.4}$$

At each step, the strategy is to focus on one position, called the ***pivot position,*** and to eliminate all terms below this position using the three elementary operations. The coefficient in the pivot position is called a ***pivotal element*** (or simply a ***pivot***), while the equation in which the pivot lies is referred to as the ***pivotal equation.*** Only nonzero numbers are allowed to be pivots. If a coefficient in a pivot position is ever 0, then the pivotal equation is interchanged with an equation *below* the pivotal equation to produce a nonzero pivot. (This is always possible for square systems possessing a unique solution.) Unless it is 0, the first coefficient of the first equation is taken as the first pivot. For example, the circled $\textcircled{2}$ in the system below is the pivot for the first step:

$$\begin{aligned} \textcircled{2}x + y + z &= 1, \\ 6x + 2y + z &= -1, \\ -2x + 2y + z &= 7. \end{aligned}$$

Step 1. Eliminate all terms below the first pivot.

- Subtract three times the first equation from the second so as to produce the equivalent system:

$$\begin{aligned} \textcircled{2}x + y + z &= 1, \\ - y - 2z &= -4 \quad (E_2 - 3E_1), \\ -2x + 2y + z &= 7. \end{aligned}$$

- Add the first equation to the third equation to produce the equivalent system:

$$\begin{aligned} \textcircled{2}x + y + z &= 1, \\ - y - 2z &= -4, \\ 3y + 2z &= 8 \quad (E_3 + E_1). \end{aligned}$$

Step 2. Select a new pivot.

- For the time being, select a new pivot by moving down and to the right.[2] If this coefficient is not 0, then it is the next pivot. Otherwise, interchange with an equation *below* this position so as to bring a nonzero number into this pivotal position. In our example, -1 is the second pivot as identified below:

$$\begin{aligned} 2x \;+\; y \;+\; z &= 1, \\ \boxed{-1}y \;-\; 2z &= -4, \\ 3y \;+\; 2z &= 8. \end{aligned}$$

Step 3. Eliminate all terms below the second pivot.

- Add three times the second equation to the third equation so as to produce the equivalent system:

$$\begin{aligned} 2x \;+\; y \;+\; z &= 1, \\ \boxed{-1}y \;-\; 2z &= -4, \\ -\;4z &= -4 \qquad (E_3 + 3E_2). \end{aligned} \tag{1.2.5}$$

- In general, at each step you move down and to the right to select the next pivot, then eliminate all terms below the pivot until you can no longer proceed. In this example, the third pivot is -4, but since there is nothing below the third pivot to eliminate, the process is complete.

At this point, we say that the system has been ***triangularized.*** A triangular system is easily solved by a simple method known as ***back substitution*** in which the last equation is solved for the value of the last unknown and then substituted back into the penultimate equation, which is in turn solved for the penultimate unknown, etc., until each unknown has been determined. For our example, solve the last equation in (1.2.5) to obtain

$$z = 1.$$

Substitute $z = 1$ back into the second equation in (1.2.5) and determine

$$y = 4 - 2z = 4 - 2(1) = 2.$$

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[2] The strategy of selecting pivots in numerical computation is usually a bit more complicated than simply using the next coefficient that is down and to the right. Use the down-and-right strategy for now, and later more practical strategies will be discussed.

Finally, substitute $z = 1$ and $y = 2$ back into the first equation in (1.2.5) to get

$$x = \frac{1}{2}(1 - y - z) = \frac{1}{2}(1 - 2 - 1) = -1,$$

which completes the solution.

It should be clear that there is no reason to write down the symbols such as "x," "y," "z," and "$=$" at each step since we are only manipulating the coefficients. If such symbols are discarded, then a system of linear equations reduces to a rectangular array of numbers in which each horizontal line represents one equation. For example, the system in (1.2.4) reduces to the following array:

$$\left(\begin{array}{rrr|r} 2 & 1 & 1 & 1 \\ 6 & 2 & 1 & -1 \\ -2 & 2 & 1 & 7 \end{array}\right). \qquad \text{(The line emphasizes where = appeared.)}$$

The array of coefficients—the numbers on the left-hand side of the vertical line—is called the ***coefficient matrix*** for the system. The entire array—the coefficient matrix augmented by the numbers from the right-hand side of the system—is called the ***augmented matrix*** associated with the system. If the coefficient matrix is denoted by $\mathbf{A}$ and the right-hand side is denoted by $\mathbf{b}$, then the augmented matrix associated with the system is denoted by $[\mathbf{A}|\mathbf{b}]$.

Formally, a ***scalar*** is either a real number or a complex number, and a ***matrix*** is a rectangular array of scalars. It is common practice to use uppercase boldface letters to denote matrices and to use the corresponding lowercase letters with two subscripts to denote individual entries in a matrix. For example,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The first subscript on an individual entry in a matrix designates the ***row*** (the horizontal line), and the second subscript denotes the ***column*** (the vertical line) that the entry occupies. For example, if

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 & 4 \\ 8 & 6 & 5 & -9 \\ -3 & 8 & 3 & 7 \end{pmatrix}, \quad \text{then} \quad a_{11} = 2,\ a_{12} = 1,\ \ldots,\ a_{34} = 7. \tag{1.2.6}$$

A ***submatrix*** of a given matrix $\mathbf{A}$ is an array obtained by deleting any combination of rows and columns from $\mathbf{A}$. For example, $\mathbf{B} = \begin{pmatrix} 2 & 4 \\ -3 & 7 \end{pmatrix}$ is a submatrix of the matrix $\mathbf{A}$ in (1.2.6) because $\mathbf{B}$ is the result of deleting the second row and the second and third columns of $\mathbf{A}$.

Matrix $\mathbf{A}$ is said to have ***shape*** or ***size*** $m \times n$—pronounced "m by n"—whenever $\mathbf{A}$ has exactly m rows and n columns. For example, the matrix in (1.2.6) is a 3×4 matrix. By agreement, 1×1 matrices are identified with scalars and vice versa. To emphasize that matrix $\mathbf{A}$ has shape $m \times n$, subscripts are sometimes placed on $\mathbf{A}$ as $\mathbf{A}_{m\times n}$. Whenever $m = n$ (i.e., when $\mathbf{A}$ has the same number of rows as columns), $\mathbf{A}$ is called a ***square matrix.*** Otherwise, $\mathbf{A}$ is said to be ***rectangular.*** Matrices consisting of a single row or a single column are often called ***row vectors*** or ***column vectors***, respectively.

The symbol $\mathbf{A}_{i*}$ is used to denote the i^{th} row, while $\mathbf{A}_{*j}$ denotes the j^{th} column of matrix $\mathbf{A}$. For example, if $\mathbf{A}$ is the matrix in (1.2.6), then

$$\mathbf{A}_{2*} = \begin{pmatrix} 8 & 6 & 5 & -9 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_{*2} = \begin{pmatrix} 1 \\ 6 \\ 8 \end{pmatrix}.$$

For a linear system of equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m, \end{array}$$

Gaussian elimination can be executed on the associated augmented matrix $[\mathbf{A}|\mathbf{b}]$ by performing elementary operations to the rows of $[\mathbf{A}|\mathbf{b}]$. These row operations correspond to the three elementary operations (1.2.1), (1.2.2), and (1.2.3) used to manipulate linear systems. For an $m \times n$ matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{1*} \\ \vdots \\ \mathbf{M}_{i*} \\ \vdots \\ \mathbf{M}_{j*} \\ \vdots \\ \mathbf{M}_{m*} \end{pmatrix},$$

the three types of ***elementary row operations*** on $\mathbf{M}$ are as follows.

- Type I: Interchange rows i and j to produce $$\begin{pmatrix} \mathbf{M}_{1*} \\ \vdots \\ \mathbf{M}_{j*} \\ \vdots \\ \mathbf{M}_{i*} \\ \vdots \\ \mathbf{M}_{m*} \end{pmatrix}. \tag{1.2.7}$$

- Type II: Replace row i by a nonzero multiple of itself to produce

$$\begin{pmatrix} \mathbf{M}_{1*} \\ \vdots \\ \alpha\mathbf{M}_{i*} \\ \vdots \\ \mathbf{M}_{m*} \end{pmatrix}, \quad \text{where} \quad \alpha \neq 0. \tag{1.2.8}$$

- Type III: Replace row j by a combination of itself plus a multiple of row i to produce

$$\begin{pmatrix} \mathbf{M}_{1*} \\ \vdots \\ \mathbf{M}_{i*} \\ \vdots \\ \mathbf{M}_{j*} + \alpha\mathbf{M}_{i*} \\ \vdots \\ \mathbf{M}_{m*} \end{pmatrix}. \tag{1.2.9}$$

To solve the system (1.2.4) by using elementary row operations, start with the associated augmented matrix $[\mathbf{A}|\mathbf{b}]$ and triangularize the coefficient matrix $\mathbf{A}$ by performing exactly the same sequence of row operations that corresponds to the elementary operations executed on the equations themselves:

$$\left(\begin{array}{rrr|r} \textcircled{2} & 1 & 1 & 1 \\ 6 & 2 & 1 & -1 \\ -2 & 2 & 1 & 7 \end{array}\right) \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + R_1 \end{array} \longrightarrow \left(\begin{array}{rrr|r} 2 & 1 & 1 & 1 \\ 0 & \textcircled{-1} & -2 & -4 \\ 0 & 3 & 2 & 8 \end{array}\right) \begin{array}{l} \\ \\ R_3 + 3R_2 \end{array}$$

$$\longrightarrow \left(\begin{array}{rrr|r} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \end{array}\right).$$

The final array represents the triangular system

$$\begin{aligned} 2x + y + z &= 1, \\ - y - 2z &= -4, \\ - 4z &= -4 \end{aligned}$$

that is solved by back substitution as described earlier. In general, if an $n \times n$ system has been triangularized to the form

$$\left(\begin{array}{cccc|c} t_{11} & t_{12} & \cdots & t_{1n} & c_1 \\ 0 & t_{22} & \cdots & t_{2n} & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{nn} & c_n \end{array}\right) \tag{1.2.10}$$

in which each $t_{ii} \neq 0$ (i.e., there are no zero pivots), then the general algorithm for back substitution is as follows.

Algorithm for Back Substitution

Determine the x_i 's from (1.2.10) by first setting $x_n = c_n/t_{nn}$ and then recursively computing

$$x_i = \frac{1}{t_{ii}}\left(c_i - t_{i,i+1}x_{i+1} - t_{i,i+2}x_{i+2} - \cdots - t_{in}x_n\right)$$

for $i = n-1,\ n-2, \ldots, 2,\ 1.$

One way to gauge the efficiency of an algorithm is to count the number of arithmetical operations required.[3] For a variety of reasons, no distinction is made between additions and subtractions, and no distinction is made between multiplications and divisions. Furthermore, multiplications/divisions are usually counted separately from additions/subtractions. Even if you do not work through the details, it is important that you be aware of the operational counts for Gaussian elimination with back substitution so that you will have a basis for comparison when other algorithms are encountered.

Gaussian Elimination Operation Counts

Gaussian elimination with back substitution applied to an $n \times n$ system requires

$$\frac{n^3}{3} + n^2 - \frac{n}{3} \quad \text{multiplications/divisions}$$

and

$$\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6} \quad \text{additions/subtractions.}$$

As n grows, the $n^3/3$ term dominates each of these expressions. Therefore, the important thing to remember is that Gaussian elimination with back substitution on an $n \times n$ system requires about $n^3/3$ multiplications/divisions and about the same number of additions/subtractions.

[3] Operation counts alone may no longer be as important as they once were in gauging the efficiency of an algorithm. Older computers executed instructions sequentially, whereas some contemporary machines are capable of executing instructions in parallel so that different numerical tasks can be performed simultaneously. An algorithm that lends itself to parallelism may have a higher operational count but might nevertheless run faster on a parallel machine than an algorithm with a lesser operational count that cannot take advantage of parallelism.

Example 1.2.1

Problem: Solve the following system using Gaussian elimination with back substitution:

$$\begin{aligned} v - w &= 3,\\ -2u + 4v - w &= 1,\\ -2u + 5v - 4w &= -2. \end{aligned}$$

Solution: The associated augmented matrix is

$$\left(\begin{array}{rrr|r} 0 & 1 & -1 & 3\\ -2 & 4 & -1 & 1\\ -2 & 5 & -4 & -2 \end{array}\right).$$

Since the first pivotal position contains 0, interchange rows one and two before eliminating below the first pivot:

$$\left(\begin{array}{rrr|r} \textcircled{0} & 1 & -1 & 3\\ -2 & 4 & -1 & 1\\ -2 & 5 & -4 & -2 \end{array}\right) \xrightarrow{\textit{Interchange } R_1 \textit{ and } R_2} \left(\begin{array}{rrr|r} \textcircled{-2} & 4 & -1 & 1\\ 0 & 1 & -1 & 3\\ -2 & 5 & -4 & -2 \end{array}\right) R_3 - R_1$$

$$\longrightarrow \left(\begin{array}{rrr|r} -2 & 4 & -1 & 1\\ 0 & \textcircled{1} & -1 & 3\\ 0 & 1 & -3 & -3 \end{array}\right) R_3 - R_2 \longrightarrow \left(\begin{array}{rrr|r} -2 & 4 & -1 & 1\\ 0 & 1 & -1 & 3\\ 0 & 0 & -2 & -6 \end{array}\right).$$

Back substitution yields

$$\begin{aligned} w &= \frac{-6}{-2} = 3,\\ v &= 3 + w = 3 + 3 = 6,\\ u &= \frac{1}{-2}(1 - 4v + w) = \frac{1}{-2}(1 - 24 + 3) = 10. \end{aligned}$$

Exercises for section 1.2

1.2.1. Use Gaussian elimination with back substitution to solve the following system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1,\\ x_1 + 2x_2 + 2x_3 &= 1,\\ x_1 + 2x_2 + 3x_3 &= 1. \end{aligned}$$

1.2.2. Apply Gaussian elimination with back substitution to the following system:

$$\begin{aligned} 2x_1 - x_2 \phantom{{}-x_3} &= 0, \\ -x_1 + 2x_2 - x_3 &= 0, \\ -x_2 + x_3 &= 1. \end{aligned}$$

1.2.3. Use Gaussian elimination with back substitution to solve the following system:

$$\begin{aligned} 4x_2 - 3x_3 &= 3, \\ -x_1 + 7x_2 - 5x_3 &= 4, \\ -x_1 + 8x_2 - 6x_3 &= 5. \end{aligned}$$

1.2.4. Solve the following system:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1, \\ x_1 + x_2 + 3x_3 + 3x_4 &= 3, \\ x_1 + x_2 + 2x_3 + 3x_4 &= 3, \\ x_1 + 3x_2 + 3x_3 + 3x_4 &= 4. \end{aligned}$$

1.2.5. Consider the following three systems where the coefficients are the same for each system, but the right-hand sides are different (this situation occurs frequently):

$$\begin{aligned} 4x - 8y + 5z &= 1 \mid 0 \mid 0, \\ 4x - 7y + 4z &= 0 \mid 1 \mid 0, \\ 3x - 4y + 2z &= 0 \mid 0 \mid 1. \end{aligned}$$

Solve all three systems at one time by performing Gaussian elimination on an augmented matrix of the form

$$\left[\mathbf{A} \mid \mathbf{b_1} \mid \mathbf{b_2} \mid \mathbf{b_3}\right].$$

1.2.6. Suppose that matrix $\mathbf{B}$ is obtained by performing a sequence of row operations on matrix $\mathbf{A}$. Explain why $\mathbf{A}$ can be obtained by performing row operations on $\mathbf{B}$.

1.2.7. Find angles α, β, and γ such that

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3, \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 2, \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9, \end{aligned}$$

where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma < \pi$.

1.2.8. The following system has no solution:

$$\begin{aligned}-x_1 + 3x_2 - 2x_3 &= 1,\\ -x_1 + 4x_2 - 3x_3 &= 0,\\ -x_1 + 5x_2 - 4x_3 &= 0.\end{aligned}$$

Attempt to solve this system using Gaussian elimination and explain what occurs to indicate that the system is impossible to solve.

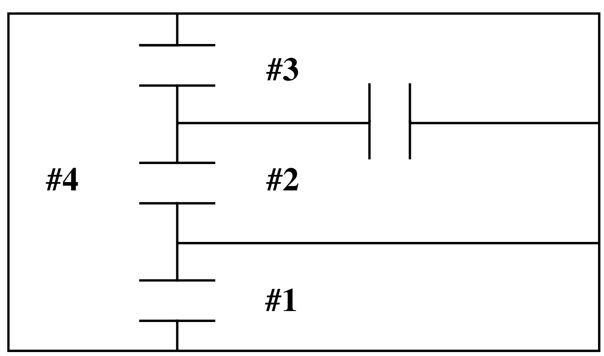
1.2.9. Attempt to solve the system

$$\begin{aligned}-x_1 + 3x_2 - 2x_3 &= 4,\\ -x_1 + 4x_2 - 3x_3 &= 5,\\ -x_1 + 5x_2 - 4x_3 &= 6,\end{aligned}$$

using Gaussian elimination and explain why this system must have infinitely many solutions.

1.2.10. By solving a 3×3 system, find the coefficients in the equation of the parabola $y = \alpha + \beta x + \gamma x^2$ that passes through the points $(1, 1)$, $(2, 2)$, and $(3, 0)$.

1.2.11. Suppose that 100 insects are distributed in an enclosure consisting of four chambers with passageways between them as shown below.



At the end of one minute, the insects have redistributed themselves. Assume that a minute is not enough time for an insect to visit more than one chamber and that at the end of a minute 40% of the insects in each chamber have not left the chamber they occupied at the beginning of the minute. The insects that leave a chamber disperse uniformly among the chambers that are directly accessible from the one they initially occupied—e.g., from #3, half move to #2 and half move to #4.

(a) If at the end of one minute there are 12, 25, 26, and 37 insects in chambers #1, #2, #3, and #4, respectively, determine what the initial distribution had to be.

(b) If the initial distribution is 20, 20, 20, 40, what is the distribution at the end of one minute?

1.2.12. Show that the three types of elementary row operations discussed on p. 8 are not independent by showing that the interchange operation (1.2.7) can be accomplished by a sequence of the other two types of row operations given in (1.2.8) and (1.2.9).

1.2.13. Suppose that $[\mathbf{A}|\mathbf{b}]$ is the augmented matrix associated with a linear system. You know that performing row operations on $[\mathbf{A}|\mathbf{b}]$ does not change the solution of the system. However, no mention of *column operations* was ever made because column operations can alter the solution.

(a) Describe the effect on the solution of a linear system when columns $\mathbf{A}_{*j}$ and $\mathbf{A}_{*k}$ are interchanged.

(b) Describe the effect when column $\mathbf{A}_{*j}$ is replaced by $\alpha\mathbf{A}_{*j}$ for $\alpha \neq 0$.

(c) Describe the effect when $\mathbf{A}_{*j}$ is replaced by $\mathbf{A}_{*j} + \alpha\mathbf{A}_{*k}$.
Hint: Experiment with a 2×2 or 3×3 system.

1.2.14. Consider the $n \times n$ ***Hilbert matrix*** defined by

$$\mathbf{H} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix}.$$

Express the individual entries h_{ij} in terms of i and j.

1.2.15. Verify that the operation counts given in the text for Gaussian elimination with back substitution are correct for a general 3×3 system. If you are up to the challenge, try to verify these counts for a general $n \times n$ system.

1.2.16. Explain why a linear system can never have exactly two different solutions. Extend your argument to explain the fact that if a system has more than one solution, then it must have infinitely many different solutions.

Solutions for Chapter 1

Solutions for exercises in section 1. 2

1.2.1. $(1, 0, 0)$

1.2.2. $(1, 2, 3)$

1.2.3. $(1, 0, -1)$

1.2.4. $(-1/2, 1/2, 0, 1)$

1.2.5. $\left(\begin{array}{c|c|c} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{array}\right)$

1.2.6. Every row operation is reversible. In particular the "inverse" of any row operation is again a row operation of the same type.

1.2.7. $\frac{\pi}{2}, \pi, 0$

1.2.8. The third equation in the triangularized form is $0x_3 = 1$, which is impossible to solve.

1.2.9. The third equation in the triangularized form is $0x_3 = 0$, and all numbers are solutions. This means that you can start the back substitution with any value whatsoever and consequently produce infinitely many solutions for the system.

1.2.10. $\alpha = -3$, $\beta = \frac{11}{2}$, and $\gamma = -\frac{3}{2}$

1.2.11. (a) If $x_i =$ the number initially in chamber $\#i$, then

$$\begin{aligned} .4x_1 + 0x_2 + 0x_3 + .2x_4 &= 12 \\ 0x_1 + .4x_2 + .3x_3 + .2x_4 &= 25 \\ 0x_1 + .3x_2 + .4x_3 + .2x_4 &= 26 \\ .6x_1 + .3x_2 + .3x_3 + .4x_4 &= 37 \end{aligned}$$

and the solution is $x_1 = 10$, $x_2 = 20$, $x_3 = 30$, and $x_4 = 40$.

(b) 16, 22, 22, 40

1.2.12. To interchange rows i and j, perform the following sequence of Type II and Type III operations.

$$\begin{array}{ll} R_j \leftarrow R_j + R_i & \text{(replace row } j \text{ by the sum of row } j \text{ and } i) \\ R_i \leftarrow R_i - R_j & \text{(replace row } i \text{ by the difference of row } i \text{ and } j) \\ R_j \leftarrow R_j + R_i & \text{(replace row } j \text{ by the sum of row } j \text{ and } i) \\ R_i \leftarrow -R_i & \text{(replace row } i \text{ by its negative)} \end{array}$$

1.2.13. (a) This has the effect of interchanging the order of the unknowns—x_j and x_k are permuted. (b) The solution to the new system is the same as the solution to the old system except that the solution for the j^{th} unknown of the new system is $\hat{x}_j = \frac{1}{\alpha}x_j$. This has the effect of "changing the units" of the j^{th} unknown. (c) The solution to the new system is the same as the solution for the old system except that the solution for the k^{th} unknown in the new system is $\hat{x}_k = x_k - \alpha x_j$.

1.2.14. $h_{ij} = \frac{1}{i+j-1}$

1.2.16. If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ are two different solutions, then

$$\mathbf{z} = \frac{\mathbf{x}+\mathbf{y}}{2} = \begin{pmatrix} \frac{x_1+y_1}{2} \\ \frac{x_2+y_2}{2} \\ \vdots \\ \frac{x_m+y_m}{2} \end{pmatrix}$$

is a third solution different from both $\mathbf{x}$ and $\mathbf{y}$.

Solutions for exercises in section 1. 3

1.3.1. $(1, 0, -1)$

1.3.2. $(2, -1, 0, 0)$

1.3.3. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

Solutions for exercises in section 1. 4

1.4.2. Use $y'(t_k) = y'_k \approx \dfrac{y_{k+1} - y_{k-1}}{2h}$ and $y''(t_k) = y''_k \approx \dfrac{y_{k-1} - 2y_k + y_{k+1}}{h^2}$ to write

$$f(t_k) = f_k = y''_k - y'_k \approx \frac{2y_{k-1} - 4y_k + 2y_{k+1}}{2h^2} - \frac{hy_{k+1} - hy_{k-1}}{2h^2}, \quad k = 1, 2, \ldots, n,$$

with $y_0 = y_{n+1} = 0$. These discrete approximations form the tridiagonal system

$$\begin{pmatrix} -4 & 2-h & & & \\ 2+h & -4 & 2-h & & \\ & \ddots & \ddots & \ddots & \\ & & 2+h & -4 & 2-h \\ & & & 2+h & -4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = 2h^2 \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix}.$$