# 1.3 GAUSS–JORDAN METHOD

The purpose of this section is to introduce a variation of Gaussian elimination that is known as the ***Gauss–Jordan method.*** [4] The two features that distinguish the Gauss–Jordan method from standard Gaussian elimination are as follows.

- At each step, the pivot element is forced to be $1$.
- At each step, all terms *above* the pivot as well as all terms below the pivot are eliminated.

In other words, if

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right)$$

is the augmented matrix associated with a linear system, then elementary row operations are used to reduce this matrix to

$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & s_1 \\ 0 & 1 & \cdots & 0 & s_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & s_n \end{array}\right).$$

The solution then appears in the last column (i.e., $x_i = s_i$) so that this procedure circumvents the need to perform back substitution.

## Example 1.3.1

**Problem:** Apply the Gauss–Jordan method to solve the following system:

$$\begin{aligned} 2x_1 + 2x_2 + 6x_3 &= 4, \\ 2x_1 + x_2 + 7x_3 &= 6, \\ -2x_1 - 6x_2 - 7x_3 &= -1. \end{aligned}$$

[4] Although there has been some confusion as to which Jordan should receive credit for this algorithm, it now seems clear that the method was in fact introduced by a geodesist named Wilhelm Jordan (1842–1899) and not by the more well known mathematician Marie Ennemond Camille Jordan (1838–1922), whose name is often mistakenly associated with the technique, but who is otherwise correctly credited with other important topics in matrix analysis, the “Jordan canonical form” being the most notable. Wilhelm Jordan was born in southern Germany, educated in Stuttgart, and was a professor of geodesy at the technical college in Karlsruhe. He was a prolific writer, and he introduced his elimination scheme in the 1888 publication *Handbuch der Vermessungskunde.* Interestingly, a method similar to W. Jordan’s variation of Gaussian elimination seems to have been discovered and described independently by an obscure Frenchman named Clasen, who appears to have published only one scientific article, which appeared in 1888—the same year as W. Jordan’s *Handbuch* appeared.

**Solution:** The sequence of operations is indicated in parentheses and the pivots are circled.

$$\left(\begin{array}{ccc|c} 2 & 2 & 6 & 4 \\ 2 & 1 & 7 & 6 \\ -2 & -6 & -7 & -1 \end{array}\right) R_1/2 \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 2 & 1 & 7 & 6 \\ -2 & -6 & -7 & -1 \end{array}\right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 2R_1 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & -4 & -1 & 3 \end{array}\right) (-R_2) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & -4 & -1 & 3 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 4R_2 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 4 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -5 & -5 \end{array}\right) -R_3/5 \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 4 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} R_1 - 4R_3 \\ R_2 + R_3 \\ \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

Therefore, the solution is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$

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On the surface it may seem that there is little difference between the Gauss–Jordan method and Gaussian elimination with back substitution because eliminating terms above the pivot with Gauss–Jordan seems equivalent to performing back substitution. But this is not correct. Gauss–Jordan requires more arithmetic than Gaussian elimination with back substitution.

## Gauss–Jordan Operation Counts

For an $n \times n$ system, the Gauss–Jordan procedure requires

$$\frac{n^3}{2} + \frac{n^2}{2} \quad \text{multiplications/divisions}$$

and

$$\frac{n^3}{2} - \frac{n}{2} \quad \text{additions/subtractions.}$$

In other words, the Gauss–Jordan method requires about $n^3/2$ multiplications/divisions and about the same number of additions/subtractions.

Recall from the previous section that Gaussian elimination with back substitution requires only about $n^3/3$ multiplications/divisions and about the same number of additions/subtractions. Compare this with the $n^3/2$ factor required by the Gauss–Jordan method, and you can see that Gauss–Jordan requires about 50% *more* effort than Gaussian elimination with back substitution. For small systems of the textbook variety (e.g., $n = 3$ ), these comparisons do not show a great deal of difference. However, in practical work, the systems that are encountered can be quite large, and the difference between Gauss–Jordan and Gaussian elimination with back substitution can be significant. For example, if $n = 100$, then $n^3/3$ is about 333,333, while $n^3/2$ is 500,000, which is a difference of 166,667 multiplications/divisions as well as that many additions/subtractions.

Although the Gauss–Jordan method is not recommended for solving linear systems that arise in practical applications, it does have some theoretical advantages. Furthermore, it can be a useful technique for tasks other than computing solutions to linear systems. We will make use of the Gauss–Jordan procedure when matrix inversion is discussed—this is the primary reason for introducing Gauss–Jordan.

## Exercises for section 1.3

**1.3.1.** Use the Gauss–Jordan method to solve the following system:

$$\begin{aligned} 4x_2 - 3x_3 &= 3, \\ -x_1 + 7x_2 - 5x_3 &= 4, \\ -x_1 + 8x_2 - 6x_3 &= 5. \end{aligned}$$

**1.3.2.** Apply the Gauss–Jordan method to the following system:

$$\begin{aligned} x_1 + \phantom{2}x_2 + \phantom{2}x_3 + \phantom{2}x_4 &= 1, \\ x_1 + 2x_2 + 2x_3 + 2x_4 &= 0, \\ x_1 + 2x_2 + 3x_3 + 3x_4 &= 0, \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 0. \end{aligned}$$

**1.3.3.** Use the Gauss–Jordan method to solve the following three systems at the same time.

$$\begin{array}{rcl|c|l} 2x_1 - x_2 \phantom{{}- x_3} &=& 1 & 0 & 0, \\ -x_1 + 2x_2 - x_3 &=& 0 & 1 & 0, \\ -x_2 + x_3 &=& 0 & 0 & 1. \end{array}$$

**1.3.4.** Verify that the operation counts given in the text for the Gauss–Jordan method are correct for a general $3 \times 3$ system. If you are up to the challenge, try to verify these counts for a general $n \times n$ system.

solution to the old system except that the solution for the $j^{th}$ unknown of the new system is $\hat{x}_j = \frac{1}{\alpha} x_j$. This has the effect of "changing the units" of the $j^{th}$ unknown. (c) The solution to the new system is the same as the solution for the old system except that the solution for the $k^{th}$ unknown in the new system is $\hat{x}_k = x_k - \alpha x_j$.

**1.2.14.** $h_{ij} = \frac{1}{i+j-1}$

**1.2.16.** If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ are two different solutions, then

$$\mathbf{z} = \frac{\mathbf{x} + \mathbf{y}}{2} = \begin{pmatrix} \frac{x_1+y_1}{2} \\ \frac{x_2+y_2}{2} \\ \vdots \\ \frac{x_m+y_m}{2} \end{pmatrix}$$

is a third solution different from both $\mathbf{x}$ and $\mathbf{y}$.

## Solutions for exercises in section 1. 3

**1.3.1.** $(1, 0, -1)$

**1.3.2.** $(2, -1, 0, 0)$

**1.3.3.** $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

## Solutions for exercises in section 1. 4

**1.4.2.** Use $y'(t_k) = y'_k \approx \dfrac{y_{k+1} - y_{k-1}}{2h}$ and $y''(t_k) = y''_k \approx \dfrac{y_{k-1} - 2y_k + y_{k+1}}{h^2}$ to write

$$f(t_k) = f_k = y''_k - y'_k \approx \frac{2y_{k-1} - 4y_k + 2y_{k+1}}{2h^2} - \frac{hy_{k+1} - hy_{k-1}}{2h^2}, \quad k = 1, 2, \ldots, n,$$

with $y_0 = y_{n+1} = 0$. These discrete approximations form the tridiagonal system

$$\begin{pmatrix} -4 & 2-h & & & \\ 2+h & -4 & 2-h & & \\ & \ddots & \ddots & \ddots & \\ & & 2+h & -4 & 2-h \\ & & & 2+h & -4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = 2h^2 \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix}.$$