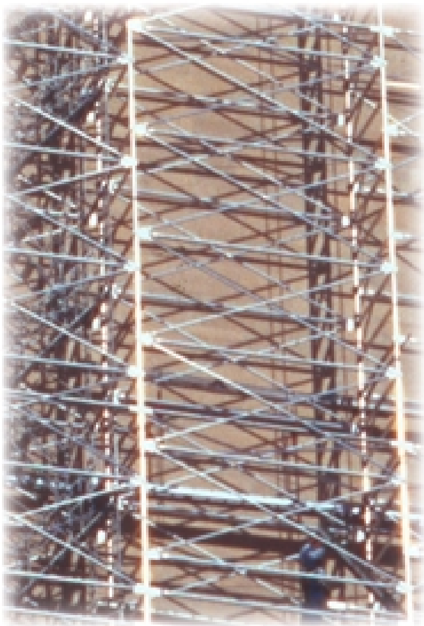

CHAPTER 2

Rectangular Systems and Echelon Forms

2.1 ROW ECHELON FORM AND RANK

We are now ready to analyze more general linear systems consisting of m linear equations involving n unknowns

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 &+ a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ &\vdots \\ a_{m1}x_1 &+ a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{aligned}$$

where m may be different from n. If we do not know for sure that m and n are the same, then the system is said to be ***rectangular.*** The case $m = n$ is still allowed in the discussion—statements concerning rectangular systems also are valid for the special case of square systems.

The first goal is to extend the Gaussian elimination technique from square systems to completely general rectangular systems. Recall that for a square system with a unique solution, the pivotal positions are always located along the ***main diagonal***—the diagonal line from the upper-left-hand corner to the lower-right-hand corner—in the coefficient matrix $\mathbf{A}$ so that Gaussian elimination results in a reduction of $\mathbf{A}$ to a ***triangular matrix***, such as that illustrated below for the case $n = 4$:

$$\mathbf{T} = \begin{pmatrix} \circledast & * & * & * \\ 0 & \circledast & * & * \\ 0 & 0 & \circledast & * \\ 0 & 0 & 0 & \circledast \end{pmatrix}.$$

Remember that a pivot must always be a nonzero number. For square systems possessing a unique solution, it is a fact (proven later) that one can always bring a nonzero number into each pivotal position along the main diagonal.[8] However, in the case of a general rectangular system, it is not always possible to have the pivotal positions lying on a straight diagonal line in the coefficient matrix. This means that the final result of Gaussian elimination will *not* be triangular in form. For example, consider the following system:

$$\begin{aligned} x_1 + 2x_2 + x_3 + 3x_4 + 3x_5 &= 5, \\ 2x_1 + 4x_2 \phantom{{}+3x_3} + 4x_4 + 4x_5 &= 6, \\ x_1 + 2x_2 + 3x_3 + 5x_4 + 5x_5 &= 9, \\ 2x_1 + 4x_2 \phantom{{}+3x_3} + 4x_4 + 7x_5 &= 9. \end{aligned}$$

Focus your attention on the coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix}, \tag{2.1.1}$$

and ignore the right-hand side for a moment. Applying Gaussian elimination to $\mathbf{A}$ yields the following result:

$$\begin{pmatrix} \textcircled{1} & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & \textcircled{0} & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix}.$$

In the basic elimination process, the strategy is to move down and to the right to the next pivotal position. If a zero occurs in this position, an interchange with a row below the pivotal row is executed so as to bring a nonzero number into the pivotal position. However, in this example, it is clearly impossible to bring a nonzero number into the $(2,2)$-position by interchanging the second row with a lower row.

In order to handle this situation, the elimination process is modified as follows.

[8] This discussion is for exact arithmetic. If floating-point arithmetic is used, this may no longer be true. Part (a) of Exercise 1.6.1 is one such example.

Modified Gaussian Elimination

Suppose that $\mathbf{U}$ is the augmented matrix associated with the system after $i-1$ elimination steps have been completed. To execute the i^{th} step, proceed as follows:

- Moving from left to right in $\mathbf{U}$, locate the first column that contains a nonzero entry on or below the i^{th} position—say it is $\mathbf{U}_{*j}$.
- The pivotal position for the i^{th} step is the (i,j)-position.
- If necessary, interchange the i^{th} row with a lower row to bring a nonzero number into the (i,j)-position, and then annihilate all entries below this pivot.
- If row $\mathbf{U}_{i*}$ as well as all rows in $\mathbf{U}$ below $\mathbf{U}_{i*}$ consist entirely of zeros, then the elimination process is completed.

Illustrated below is the result of applying this modified version of Gaussian elimination to the matrix given in (2.1.1).

Example 2.1.1

Problem: Apply modified Gaussian elimination to the following matrix and circle the pivot positions:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} \textcircled{1} & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 1 & 3 & 3 \\ 0 & 0 & \textcircled{-2} & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 1 & 3 & 3 \\ 0 & 0 & \textcircled{-2} & -2 & -2 \\ 0 & 0 & 0 & 0 & \textcircled{0} \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 1 & 3 & 3 \\ 0 & 0 & \textcircled{-2} & -2 & -2 \\ 0 & 0 & 0 & 0 & \textcircled{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that the final result of applying Gaussian elimination in the above example is not a purely triangular form but rather a jagged or "stair-step" type of triangular form. Hereafter, a matrix that exhibits this stair-step structure will be said to be in ***row echelon form.***

Row Echelon Form

An $m \times n$ matrix $\mathbf{E}$ with rows $\mathbf{E}_{i*}$ and columns $\mathbf{E}_{*j}$ is said to be in ***row echelon form*** provided the following two conditions hold.

- If $\mathbf{E}_{i*}$ consists entirely of zeros, then all rows below $\mathbf{E}_{i*}$ are also entirely zero; i.e., all zero rows are at the bottom.

- If the first nonzero entry in $\mathbf{E}_{i*}$ lies in the j^{th} position, then all entries below the i^{th} position in columns $\mathbf{E}_{*1}, \mathbf{E}_{*2}, \ldots, \mathbf{E}_{*j}$ are zero.

These two conditions say that the nonzero entries in an echelon form must lie on or above a stair-step line that emanates from the upper-left-hand corner and slopes down and to the right. The pivots are the first nonzero entries in each row. A typical structure for a matrix in row echelon form is illustrated below with the pivots circled.

$$\begin{pmatrix} \circledast & * & * & * & * & * & * & * \\ 0 & 0 & \circledast & * & * & * & * & * \\ 0 & 0 & 0 & \circledast & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \circledast & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Because of the flexibility in choosing row operations to reduce a matrix $\mathbf{A}$ to a row echelon form $\mathbf{E}$, the entries in $\mathbf{E}$ are not uniquely determined by $\mathbf{A}$. Nevertheless, it can be proven that the "form" of $\mathbf{E}$ is unique in the sense that *the positions of the pivots in* $\mathbf{E}$ *(and* $\mathbf{A}$*) are uniquely determined by the entries in* $\mathbf{A}$.[9] Because the pivotal positions are unique, it follows that the number of pivots, which is the same as the number of nonzero rows in $\mathbf{E}$, is also uniquely determined by the entries in $\mathbf{A}$. This number is called the ***rank***[10] of $\mathbf{A}$, and it

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[9] The fact that the pivotal positions are unique should be intuitively evident. If it isn't, take the matrix given in (2.1.1) and try to force some different pivotal positions by a different sequence of row operations.

[10] The word "rank" was introduced in 1879 by the German mathematician Ferdinand Georg Frobenius (p. 662), who thought of it as the size of the largest nonzero minor determinant in $\mathbf{A}$. But the concept had been used as early as 1851 by the English mathematician James J. Sylvester (1814–1897).

is extremely important in the development of our subject.

Rank of a Matrix

Suppose $\mathbf{A}_{m\times n}$ is reduced by row operations to an echelon form $\mathbf{E}$. The ***rank*** of $\mathbf{A}$ is defined to be the number

$$\begin{aligned} rank\,(\mathbf{A}) &= \text{ number of pivots} \\ &= \text{ number of nonzero rows in } \mathbf{E} \\ &= \text{ number of basic columns in } \mathbf{A}, \end{aligned}$$

where the ***basic columns*** of $\mathbf{A}$ are defined to be those columns in $\mathbf{A}$ that contain the pivotal positions.

Example 2.1.2

Problem: Determine the rank, and identify the basic columns in

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{pmatrix}.$$

Solution: Reduce $\mathbf{A}$ to row echelon form as shown below:

$$\mathbf{A} = \begin{pmatrix} \textcircled{1} & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 1 & 1 \\ 0 & 0 & 0 & \textcircled{0} \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 1 & 1 \\ 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{E}.$$

Consequently, $rank\,(\mathbf{A}) = 2$. The pivotal positions lie in the first and fourth columns so that the basic columns of $\mathbf{A}$ are $\mathbf{A}_{*1}$ and $\mathbf{A}_{*4}$. That is,

$$\text{Basic Columns} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right\}.$$

Pay particular attention to the fact that the basic columns are extracted from $\mathbf{A}$ and not from the row echelon form $\mathbf{E}$.

Exercises for section 2.1

2.1.1. Reduce each of the following matrices to row echelon form, determine the rank, and identify the basic columns.

(a) $\begin{pmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \\ 2 & 6 & 0 \\ 1 & 2 & 5 \\ 3 & 8 & 6 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 1 & 1 & 3 & 0 & 4 & 1 \\ 4 & 2 & 4 & 4 & 1 & 5 & 5 \\ 2 & 1 & 3 & 1 & 0 & 4 & 3 \\ 6 & 3 & 4 & 8 & 1 & 9 & 5 \\ 0 & 0 & 3 & -3 & 0 & 0 & 3 \\ 8 & 4 & 2 & 14 & 1 & 13 & 3 \end{pmatrix}$

2.1.2. Determine which of the following matrices are in row echelon form:

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix}$. (b) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 2 & 2 & 3 & -4 \\ 0 & 0 & 7 & -8 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. (d) $\begin{pmatrix} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

2.1.3. Suppose that $\mathbf{A}$ is an $m \times n$ matrix. Give a short explanation of why each of the following statements is true.
(a) $rank\,(\mathbf{A}) \le \min\{m, n\}$.
(b) $rank\,(\mathbf{A}) < m$ if one row in $\mathbf{A}$ is entirely zero.
(c) $rank\,(\mathbf{A}) < m$ if one row in $\mathbf{A}$ is a multiple of another row.
(d) $rank\,(\mathbf{A}) < m$ if one row in $\mathbf{A}$ is a combination of other rows.
(e) $rank\,(\mathbf{A}) < n$ if one column in $\mathbf{A}$ is entirely zero.

2.1.4. Let $\mathbf{A} = \begin{pmatrix} .1 & .2 & .3 \\ .4 & .5 & .6 \\ .7 & .8 & .901 \end{pmatrix}$.
(a) Use exact arithmetic to determine $rank\,(\mathbf{A})$.
(b) Now use 3-digit floating-point arithmetic (without partial pivoting or scaling) to determine $rank\,(\mathbf{A})$. This number might be called the "3-digit numerical rank."
(c) What happens if partial pivoting is incorporated?

2.1.5. How many different "forms" are possible for a 3×4 matrix that is in row echelon form?

2.1.6. Suppose that $[\mathbf{A}|\mathbf{b}]$ is reduced to a matrix $[\mathbf{E}|\mathbf{c}]$.
(a) Is $[\mathbf{E}|\mathbf{c}]$ in row echelon form if $\mathbf{E}$ is?
(b) If $[\mathbf{E}|\mathbf{c}]$ is in row echelon form, must $\mathbf{E}$ be in row echelon form?

Solutions for Chapter 2

Solutions for exercises in section 2. 1

2.1.1. (a) $\begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ is one possible answer. Rank = 3 and the basic columns are $\{\mathbf{A}_{*1}, \mathbf{A}_{*2}, \mathbf{A}_{*4}\}$. (b) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is one possible answer. Rank = 3 and every column in $\mathbf{A}$ is basic.

(c) $\begin{pmatrix} 2 & 1 & 1 & 3 & 0 & 4 & 1 \\ 0 & 0 & 2 & -2 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is one possible answer. The rank is 3, and the basic columns are $\{\mathbf{A}_{*1}, \mathbf{A}_{*3}, \mathbf{A}_{*5}\}$.

2.1.2. (c) and (d) are in row echelon form.

2.1.3. (a) Since any row or column can contain at most one pivot, the number of pivots cannot exceed the number of rows nor the number of columns. (b) A zero row cannot contain a pivot. (c) If one row is a multiple of another, then one of them can be annihilated by the other to produce a zero row. Now the result of the previous part applies. (d) One row can be annihilated by the associated combination of row operations. (e) If a column is zero, then there are fewer than n basic columns because each basic column must contain a pivot.

2.1.4. (a) $rank\,(\mathbf{A}) = 3$ (b) 3-digit $rank\,(\mathbf{A}) = 2$ (c) With PP, 3-digit $rank\,(\mathbf{A}) = 3$

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2.1.6. (a) No, consider the form $\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \end{array}\right)$ (b) Yes—in fact, $\mathbf{E}$ is a row echelon form obtainable from $\mathbf{A}$.

Solutions for exercises in section 2. 2

2.2.1. (a) $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{A}_{*3} = 2\mathbf{A}_{*1} + \frac{1}{2}\mathbf{A}_{*2}$