2.2 REDUCED ROW ECHELON FORM

At each step of the Gauss–Jordan method, the pivot is forced to be a 1, and then all entries above and below the pivotal 1 are annihilated. If $\mathbf{A}$ is the coefficient matrix for a square system with a unique solution, then the end result of applying the Gauss–Jordan method to $\mathbf{A}$ is a matrix with 1's on the main diagonal and 0's everywhere else. That is,

$$\mathbf{A} \xrightarrow{\text{Gauss–Jordan}} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

But if the Gauss–Jordan technique is applied to a more general $m \times n$ matrix, then the final result is not necessarily the same as described above. The following example illustrates what typically happens in the rectangular case.

Example 2.2.1

Problem: Apply Gauss–Jordan elimination to the following 4×5 matrix and circle the pivot positions. This is the same matrix used in Example 2.1.1:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} \textcircled{1} & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 1 & 3 & 3 \\ 0 & 0 & \textcircled{-2} & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 1 & 3 & 3 \\ 0 & 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 0 & 2 & 2 \\ 0 & 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & \textcircled{0} \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 0 & 2 & 2 \\ 0 & 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & \textcircled{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 0 & 2 & 2 \\ 0 & 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 0 & 2 & 0 \\ 0 & 0 & \textcircled{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Compare the results of this example with the results of Example 2.1.1, and notice that the "form" of the final matrix is the same in both examples, which indeed must be the case because of the uniqueness of "form" mentioned in the previous section. The only difference is in the numerical value of some of the entries. By the nature of Gauss–Jordan elimination, each pivot is 1 and all entries above and below each pivot are 0. Consequently, the row echelon form produced by the Gauss–Jordan method contains a reduced number of nonzero entries, so it seems only natural to refer to this as a ***reduced row echelon form.***[11]

Reduced Row Echelon Form

A matrix $\mathbf{E}_{m\times n}$ is said to be in ***reduced row echelon form*** provided that the following three conditions hold.

- $\mathbf{E}$ is in row echelon form.
- The first nonzero entry in each row (i.e., each pivot) is 1.
- All entries above each pivot are 0.

A typical structure for a matrix in reduced row echelon form is illustrated below, where entries marked * can be either zero or nonzero numbers:

$$\begin{pmatrix} \textcircled{1} & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & \textcircled{1} & 0 & * & * & 0 & * \\ 0 & 0 & 0 & \textcircled{1} & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

As previously stated, if matrix $\mathbf{A}$ is transformed to a row echelon form by row operations, then the "form" is uniquely determined by $\mathbf{A}$, but the individual entries in the form are not unique. However, if $\mathbf{A}$ is transformed by row operations to a *reduced* row echelon form $\mathbf{E_A}$, then it can be shown[12] that both the "form" as well as the individual entries in $\mathbf{E_A}$ are uniquely determined by $\mathbf{A}$. In other words, the reduced row echelon form $\mathbf{E_A}$ produced from $\mathbf{A}$ is independent of whatever elimination scheme is used. Producing an unreduced form is computationally more efficient, but the uniqueness of $\mathbf{E_A}$ makes it more useful for theoretical purposes.

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[11] In some of the older books this is called the *Hermite normal form* in honor of the French mathematician Charles Hermite (1822–1901), who, around 1851, investigated reducing matrices by row operations.

[12] A formal uniqueness proof must wait until Example 3.9.2, but you can make this intuitively clear right now with some experiments. Try to produce two different reduced row echelon forms from the same matrix.

$\mathbf{E_A}$ Notation

For a matrix $\mathbf{A}$, the symbol $\mathbf{E_A}$ will hereafter denote the unique reduced row echelon form derived from $\mathbf{A}$ by means of row operations.

Example 2.2.2

Problem: Determine $\mathbf{E_A}$, deduce $rank\,(\mathbf{A})$, and identify the basic columns of

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 & 1 \\ 2 & 4 & 4 & 6 & 2 \\ 3 & 6 & 6 & 9 & 6 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} \textcircled{1} & 2 & 2 & 3 & 1 \\ 2 & 4 & 4 & 6 & 2 \\ 3 & 6 & 6 & 9 & 6 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 2 & 3 & 1 \\ 0 & 0 & \textcircled{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 2 & 3 & 1 \\ 0 & 0 & \textcircled{2} & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 2 & 3 & 1 \\ 0 & 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 0 & 1 & -1 \\ 0 & 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & \textcircled{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 0 & 1 & -1 \\ 0 & 0 & \textcircled{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \textcircled{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, $rank\,(\mathbf{A}) = 3$, and $\{\mathbf{A}_{*1}, \mathbf{A}_{*3}, \mathbf{A}_{*5}\}$ are the three basic columns.

The above example illustrates another important feature of $\mathbf{E_A}$ and explains why the basic columns are indeed "basic." Each nonbasic column is expressible as a combination of basic columns. In Example 2.2.2,

$$\mathbf{A}_{*2} = 2\mathbf{A}_{*1} \quad \text{and} \quad \mathbf{A}_{*4} = \mathbf{A}_{*1} + \mathbf{A}_{*3}. \tag{2.2.1}$$

Notice that exactly the same set of relationships hold in $\mathbf{E_A}$. That is,

$$\mathbf{E}_{*2} = 2\mathbf{E}_{*1} \quad \text{and} \quad \mathbf{E}_{*4} = \mathbf{E}_{*1} + \mathbf{E}_{*3}. \tag{2.2.2}$$

This is no coincidence—it's characteristic of what happens in general. There's more to observe. The relationships between the nonbasic and basic columns in a general matrix $\mathbf{A}$ are usually obscure, but the relationships among the columns in $\mathbf{E_A}$ are absolutely transparent. For example, notice that the multipliers used in the relationships (2.2.1) and (2.2.2) appear explicitly in the two nonbasic columns in $\mathbf{E_A}$ —they are just the nonzero entries in these nonbasic columns. This is important because it means that $\mathbf{E_A}$ can be used as a “map” or “key” to discover or unlock the hidden relationships among the columns of $\mathbf{A}$.

Finally, observe from Example 2.2.2 that only the basic columns *to the left* of a given nonbasic column are needed in order to express the nonbasic column as a combination of basic columns—e.g., representing $\mathbf{A}_{*2}$ requires only $\mathbf{A}_{*1}$ and not $\mathbf{A}_{*3}$ or $\mathbf{A}_{*5}$, while representing $\mathbf{A}_{*4}$ requires only $\mathbf{A}_{*1}$ and $\mathbf{A}_{*3}$. This too is typical. For the time being, we accept the following statements to be true. A rigorous proof is given later on p. 136.

Column Relationships in A and $\mathbf{E_A}$

- Each nonbasic column $\mathbf{E}_{*k}$ in $\mathbf{E_A}$ is a combination (a sum of multiples) of the basic columns in $\mathbf{E_A}$ to the left of $\mathbf{E}_{*k}$. That is,

$$\mathbf{E}_{*k} = \mu_1\mathbf{E}_{*b_1} + \mu_2\mathbf{E}_{*b_2} + \cdots + \mu_j\mathbf{E}_{*b_j}$$
$$= \mu_1\begin{pmatrix}1\\0\\\vdots\\0\\\vdots\\0\end{pmatrix} + \mu_2\begin{pmatrix}0\\1\\\vdots\\0\\\vdots\\0\end{pmatrix} + \cdots + \mu_j\begin{pmatrix}0\\0\\\vdots\\1\\\vdots\\0\end{pmatrix} = \begin{pmatrix}\mu_1\\\mu_2\\\vdots\\\mu_j\\\vdots\\0\end{pmatrix},$$

  where the $\mathbf{E}_{*b_i}$'s are the basic columns to the left of $\mathbf{E}_{*k}$ and where the multipliers μ_i are the first j entries in $\mathbf{E}_{*k}$.

- The relationships that exist among the columns of $\mathbf{A}$ are exactly the same as the relationships that exist among the columns of $\mathbf{E_A}$. In particular, if $\mathbf{A}_{*k}$ is a nonbasic column in $\mathbf{A}$, then

$$\mathbf{A}_{*k} = \mu_1\mathbf{A}_{*b_1} + \mu_2\mathbf{A}_{*b_2} + \cdots + \mu_j\mathbf{A}_{*b_j}, \tag{2.2.3}$$

  where the $\mathbf{A}_{*b_i}$'s are the basic columns to the left of $\mathbf{A}_{*k}$, and where the multipliers μ_i are as described above—the first j entries in $\mathbf{E}_{*k}$.

Example 2.2.3

Problem: Write each nonbasic column as a combination of basic columns in

$$\mathbf{A} = \begin{pmatrix} 2 & -4 & -8 & 6 & 3 \\ 0 & 1 & 3 & 2 & 3 \\ 3 & -2 & 0 & 0 & 8 \end{pmatrix}.$$

Solution: Transform $\mathbf{A}$ to $\mathbf{E_A}$ as shown below.

$$\begin{pmatrix} \textcircled{2} & -4 & -8 & 6 & 3 \\ 0 & 1 & 3 & 2 & 3 \\ 3 & -2 & 0 & 0 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & -2 & -4 & 3 & \frac{3}{2} \\ 0 & 1 & 3 & 2 & 3 \\ 3 & -2 & 0 & 0 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & -2 & -4 & 3 & \frac{3}{2} \\ 0 & \textcircled{1} & 3 & 2 & 3 \\ 0 & 4 & 12 & -9 & \frac{7}{2} \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} \textcircled{1} & 0 & 2 & 7 & \frac{15}{2} \\ 0 & \textcircled{1} & 3 & 2 & 3 \\ 0 & 0 & 0 & -17 & -\frac{17}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 0 & 2 & 7 & \frac{15}{2} \\ 0 & \textcircled{1} & 3 & 2 & 3 \\ 0 & 0 & 0 & \textcircled{1} & \frac{1}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} \textcircled{1} & 0 & 2 & 0 & 4 \\ 0 & \textcircled{1} & 3 & 0 & 2 \\ 0 & 0 & 0 & \textcircled{1} & \frac{1}{2} \end{pmatrix}$$

The third and fifth columns are nonbasic. Looking at the columns in $\mathbf{E_A}$ reveals

$$\mathbf{E}_{*3} = 2\mathbf{E}_{*1} + 3\mathbf{E}_{*2} \quad \text{and} \quad \mathbf{E}_{*5} = 4\mathbf{E}_{*1} + 2\mathbf{E}_{*2} + \frac{1}{2}\mathbf{E}_{*4}.$$

The relationships that exist among the columns of $\mathbf{A}$ must be exactly the same as those in $\mathbf{E_A}$, so

$$\mathbf{A}_{*3} = 2\mathbf{A}_{*1} + 3\mathbf{A}_{*2} \quad \text{and} \quad \mathbf{A}_{*5} = 4\mathbf{A}_{*1} + 2\mathbf{A}_{*2} + \frac{1}{2}\mathbf{A}_{*4}.$$

You can easily check the validity of these equations by direct calculation.

In summary, the utility of $\mathbf{E_A}$ lies in its ability to reveal dependencies in data stored as columns in an array $\mathbf{A}$. The nonbasic columns in $\mathbf{A}$ represent redundant information in the sense that this information can always be expressed in terms of the data contained in the basic columns.

Although data compression is not the primary reason for introducing $\mathbf{E_A}$, the application to these problems is clear. For a large array of data, it may be more efficient to store only "independent data" (i.e., the basic columns of $\mathbf{A}$) along with the nonzero multipliers μ_i obtained from the nonbasic columns in $\mathbf{E_A}$. Then the redundant data contained in the nonbasic columns of $\mathbf{A}$ can always be reconstructed if and when it is called for.

Exercises for section 2.2

2.2.1. Determine the reduced row echelon form for each of the following matrices and then express each nonbasic column in terms of the basic columns:

(a) $\begin{pmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 1 & 1 & 3 & 0 & 4 & 1 \\ 4 & 2 & 4 & 4 & 1 & 5 & 5 \\ 2 & 1 & 3 & 1 & 0 & 4 & 3 \\ 6 & 3 & 4 & 8 & 1 & 9 & 5 \\ 0 & 0 & 3 & -3 & 0 & 0 & 3 \\ 8 & 4 & 2 & 14 & 1 & 13 & 3 \end{pmatrix}$.

2.2.2. Construct a matrix $\mathbf{A}$ whose reduced row echelon form is

$$\mathbf{E_A} = \begin{pmatrix} 1 & 2 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Is $\mathbf{A}$ unique?

2.2.3. Suppose that $\mathbf{A}$ is an $m \times n$ matrix. Give a short explanation of why $rank\,(\mathbf{A}) < n$ whenever one column in $\mathbf{A}$ is a combination of other columns in $\mathbf{A}$.

2.2.4. Consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} .1 & .2 & .3 \\ .4 & .5 & .6 \\ .7 & .8 & .901 \end{pmatrix}.$$

(a) Use exact arithmetic to determine $\mathbf{E_A}$.
(b) Now use 3-digit floating-point arithmetic (without partial pivoting or scaling) to determine $\mathbf{E_A}$ and formulate a statement concerning "near relationships" between the columns of $\mathbf{A}$.

2.2.5. Consider the matrix

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

You already know that $\mathbf{E}_{*3}$ can be expressed in terms of $\mathbf{E}_{*1}$ and $\mathbf{E}_{*2}$. However, this is not the only way to represent the column dependencies in $\mathbf{E}$. Show how to write $\mathbf{E}_{*1}$ in terms of $\mathbf{E}_{*2}$ and $\mathbf{E}_{*3}$ and then express $\mathbf{E}_{*2}$ as a combination of $\mathbf{E}_{*1}$ and $\mathbf{E}_{*3}$. **Note:** This exercise illustrates that the set of pivotal columns is not the only set that can play the role of "basic columns." Taking the basic columns to be the ones containing the pivots is a matter of convenience because everything becomes automatic that way.

Solutions for Chapter 2

Solutions for exercises in section 2.1

2.1.1. (a) $\begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ is one possible answer. Rank = 3 and the basic columns are $\{\mathbf{A}_{*1}, \mathbf{A}_{*2}, \mathbf{A}_{*4}\}$. (b) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is one possible answer. Rank = 3 and every column in $\mathbf{A}$ is basic.

(c) $\begin{pmatrix} 2 & 1 & 1 & 3 & 0 & 4 & 1 \\ 0 & 0 & 2 & -2 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is one possible answer. The rank is 3, and the basic columns are $\{\mathbf{A}_{*1}, \mathbf{A}_{*3}, \mathbf{A}_{*5}\}$.

2.1.2. (c) and (d) are in row echelon form.

2.1.3. (a) Since any row or column can contain at most one pivot, the number of pivots cannot exceed the number of rows nor the number of columns. (b) A zero row cannot contain a pivot. (c) If one row is a multiple of another, then one of them can be annihilated by the other to produce a zero row. Now the result of the previous part applies. (d) One row can be annihilated by the associated combination of row operations. (e) If a column is zero, then there are fewer than n basic columns because each basic column must contain a pivot.

2.1.4. (a) $rank(\mathbf{A}) = 3$ (b) 3-digit $rank(\mathbf{A}) = 2$ (c) With PP, 3-digit $rank(\mathbf{A}) = 3$

2.1.5. 15

2.1.6. (a) No, consider the form $\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \end{array}\right)$ (b) Yes—in fact, $\mathbf{E}$ is a row echelon form obtainable from $\mathbf{A}$.

Solutions for exercises in section 2.2

2.2.1. (a) $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{A}_{*3} = 2\mathbf{A}_{*1} + \frac{1}{2}\mathbf{A}_{*2}$

(b) $\begin{pmatrix} 1 & \frac{1}{2} & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and

$\mathbf{A}_{*2} = \frac{1}{2}\mathbf{A}_{*1}, \quad \mathbf{A}_{*4} = 2\mathbf{A}_{*1} - \mathbf{A}_{*3}, \quad \mathbf{A}_{*6} = 2\mathbf{A}_{*1} - 3\mathbf{A}_{*5}, \quad \mathbf{A}_{*7} = \mathbf{A}_{*3} + \mathbf{A}_{*5}$

2.2.2. No.

2.2.3. The same would have to hold in $\mathbf{E}_\mathbf{A}$, and there you can see that this means not all columns can be basic. Remember, $rank\,(\mathbf{A}) =$ number of basic columns.

2.2.4. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\mathbf{A}_{*3}$ is almost a combination of $\mathbf{A}_{*1}$ and $\mathbf{A}_{*2}$. In particular, $\mathbf{A}_{*3} \approx -\mathbf{A}_{*1} + 2\mathbf{A}_{*2}$.

2.2.5. $\mathbf{E}_{*1} = 2\mathbf{E}_{*2} - \mathbf{E}_{*3}$ and $\mathbf{E}_{*2} = \frac{1}{2}\mathbf{E}_{*1} + \frac{1}{2}\mathbf{E}_{*3}$

Solutions for exercises in section 2. 3

2.3.1. (a), (b)—There is no need to do any arithmetic for this one because the right-hand side is entirely zero so that you know (0,0,0) is automatically one solution. (d), (f)

2.3.3. It is always true that $rank\,(\mathbf{A}) \le rank[\mathbf{A}|\mathbf{b}] \le m$. Since $rank\,(\mathbf{A}) = m$, it follows that $rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A})$.

2.3.4. Yes—Consistency implies that $\mathbf{b}$ and $\mathbf{c}$ are each combinations of the basic columns in $\mathbf{A}$. If $\mathbf{b} = \sum \beta_i \mathbf{A}_{*b_i}$ and $\mathbf{c} = \sum \gamma_i \mathbf{A}_{*b_i}$ where the $\mathbf{A}_{*b_i}$'s are the basic columns, then $\mathbf{b} + \mathbf{c} = \sum(\beta_i + \gamma_i)\mathbf{A}_{*b_i} = \sum \xi_i \mathbf{A}_{*b_i}$, where $\xi_i = \beta_i + \gamma_i$ so that $\mathbf{b} + \mathbf{c}$ is also a combination of the basic columns in $\mathbf{A}$.

2.3.5. Yes—because the 4×3 system $\alpha + \beta x_i + \gamma x_i^2 = y_i$ obtained by using the four given points (x_i, y_i) is consistent.

2.3.6. The system is inconsistent using 5-digits but consistent when 6-digits are used.

2.3.7. If x, y, and z denote the number of pounds of the respective brands applied, then the following constraints must be met.

$$\begin{aligned} \text{total \# units of phosphorous} &= 2x + y + z = 10 \\ \text{total \# units of potassium} &= 3x + 3y = 9 \\ \text{total \# units of nitrogen} &= 5x + 4y + z = 19 \end{aligned}$$

Since this is a consistent system, the recommendation can be satisfied exactly. Of course, the solution tells how much of each brand to apply.

2.3.8. No—if one or more such rows were ever present, how could you possibly eliminate all of them with row operations? You could eliminate all but one, but then there is no way to eliminate the last remaining one, and hence it would have to appear in the final form.