2.3 CONSISTENCY OF LINEAR SYSTEMS

A system of m linear equations in n unknowns is said to be a ***consistent*** system if it possesses at least one solution. If there are no solutions, then the system is called ***inconsistent.*** The purpose of this section is to determine conditions under which a given system will be consistent.

Stating conditions for consistency of systems involving only two or three unknowns is easy. A linear equation in two unknowns represents a line in 2-space, and a linear equation in three unknowns is a plane in 3-space. Consequently, a linear system of m equations in two unknowns is consistent if and only if the m lines defined by the m equations have at least one common point of intersection. Similarly, a system of m equations in three unknowns is consistent if and only if the associated m planes have at least one common point of intersection. However, when m is large, these geometric conditions may not be easy to verify visually, and when $n > 3$, the generalizations of intersecting lines or planes are impossible to visualize with the eye.

Rather than depending on geometry to establish consistency, we use Gaussian elimination. If the associated augmented matrix $[\mathbf{A}|\mathbf{b}]$ is reduced by row operations to a matrix $[\mathbf{E}|\mathbf{c}]$ that is in row echelon form, then consistency—or lack of it—becomes evident. Suppose that somewhere in the process of reducing $[\mathbf{A}|\mathbf{b}]$ to $[\mathbf{E}|\mathbf{c}]$ a situation arises in which the only nonzero entry in a row appears on the right-hand side, as illustrated below:

$$\text{Row } i \longrightarrow \left(\begin{array}{cccccc|c} * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}\right) \longleftarrow \alpha \neq 0.$$

If this occurs in the i^{th} row, then the i^{th} equation of the associated system is

$$0x_1 + 0x_2 + \cdots + 0x_n = \alpha.$$

For $\alpha \neq 0$, this equation has no solution, and hence the original system must also be inconsistent (because row operations don't alter the solution set). The converse also holds. That is, if a system is inconsistent, then somewhere in the elimination process a row of the form

$$\left(0 \quad 0 \quad \cdots \quad 0 \quad | \quad \alpha\right), \quad \alpha \neq 0 \tag{2.3.1}$$

must appear. Otherwise, the back substitution process can be completed and a solution is produced. There is *no* inconsistency indicated when a row of the form $(0\ 0\ \cdots\ 0\,|\,0)$ is encountered. This simply says that $0 = 0$, and although this is no help in determining the value of any unknown, it is nevertheless a true statement, so it doesn't indicate inconsistency in the system.

There are some other ways to characterize the consistency (or inconsistency) of a system. One of these is to observe that if the last column $\mathbf{b}$ in the augmented matrix $[\mathbf{A}|\mathbf{b}]$ is a nonbasic column, then no pivot can exist in the last column, and hence the system is consistent because the situation (2.3.1) cannot occur. Conversely, if the system is consistent, then the situation (2.3.1) never occurs during Gaussian elimination and consequently the last column cannot be basic. In other words, $[\mathbf{A}|\mathbf{b}]$ is consistent if and only if $\mathbf{b}$ is a nonbasic column.

Saying that $\mathbf{b}$ is a nonbasic column in $[\mathbf{A}|\mathbf{b}]$ is equivalent to saying that all basic columns in $[\mathbf{A}|\mathbf{b}]$ lie in the coefficient matrix $\mathbf{A}$. Since the number of basic columns in a matrix is the rank, consistency may also be characterized by stating that a system is consistent if and only if $rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A})$.

Recall from the previous section the fact that each nonbasic column in $[\mathbf{A}|\mathbf{b}]$ must be expressible in terms of the basic columns. Because a consistent system is characterized by the fact that the right-hand side $\mathbf{b}$ is a nonbasic column, it follows that a system is consistent if and only if the right-hand side $\mathbf{b}$ is a combination of columns from the coefficient matrix $\mathbf{A}$.

Each of the equivalent[13] ways of saying that a system is consistent is summarized below.

Consistency

Each of the following is equivalent to saying that $[\mathbf{A}|\mathbf{b}]$ is consistent.

- In row reducing $[\mathbf{A}|\mathbf{b}]$, a row of the following form never appears:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & | & \alpha \end{pmatrix}, \quad \text{where} \quad \alpha \neq 0. \tag{2.3.2}$$

- $\mathbf{b}$ is a nonbasic column in $[\mathbf{A}|\mathbf{b}]$. (2.3.3)
- $rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A})$. (2.3.4)
- $\mathbf{b}$ is a combination of the basic columns in $\mathbf{A}$. (2.3.5)

Example 2.3.1

Problem: Determine if the following system is consistent:

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 1,\\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 3x_5 &= 1,\\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 2x_5 &= 2,\\ 3x_1 + 5x_2 + 8x_3 + 6x_4 + 5x_5 &= 3. \end{aligned}$$

[13] Statements P and Q are said to be equivalent when (P implies Q) as well as its converse (Q implies P) are true statements. This is also the meaning of the phrase "P if and only if Q."

Solution: Apply Gaussian elimination to the augmented matrix $[\mathbf{A}|\mathbf{b}]$ as shown:

$$\left(\begin{array}{ccccc|c} \textcircled{1} & 1 & 2 & 2 & 1 & 1 \\ 2 & 2 & 4 & 4 & 3 & 1 \\ 2 & 2 & 4 & 4 & 2 & 2 \\ 3 & 5 & 8 & 6 & 5 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} \textcircled{1} & 1 & 2 & 2 & 1 & 1 \\ 0 & \textcircled{0} & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} \textcircled{1} & 1 & 2 & 2 & 1 & 1 \\ 0 & \textcircled{2} & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Because a row of the form $(0 \;\; 0 \;\; \cdots \;\; 0 \;\; | \;\; \alpha)$ with $\alpha \neq 0$ never emerges, the system is consistent. We might also observe that $\mathbf{b}$ is a nonbasic column in $[\mathbf{A}|\mathbf{b}]$ so that $rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A})$. Finally, by completely reducing $\mathbf{A}$ to $\mathbf{E_A}$, it is possible to verify that $\mathbf{b}$ is indeed a combination of the basic columns $\{\mathbf{A}_{*1}, \mathbf{A}_{*2}, \mathbf{A}_{*5}\}$.

Exercises for section 2.3

2.3.1. Determine which of the following systems are consistent.

(a)
$$\begin{aligned} x + 2y + z &= 2, \\ 2x + 4y \phantom{{}+z} &= 2, \\ 3x + 6y + z &= 4. \end{aligned}$$

(b)
$$\begin{aligned} 2x + 2y + 4z &= 0, \\ 3x + 2y + 5z &= 0, \\ 4x + 2y + 6z &= 0. \end{aligned}$$

(c)
$$\begin{aligned} x - y + z &= 1, \\ x - y - z &= 2, \\ x + y - z &= 3, \\ x + y + z &= 4. \end{aligned}$$

(d)
$$\begin{aligned} x - y + z &= 1, \\ x - y - z &= 2, \\ x + y - z &= 3, \\ x + y + z &= 2. \end{aligned}$$

(e)
$$\begin{aligned} 2w + x + 3y + 5z &= 1, \\ 4w + \phantom{x+{}} 4y + 8z &= 0, \\ w + x + 2y + 3z &= 0, \\ x + y + z &= 0. \end{aligned}$$

(f)
$$\begin{aligned} 2w + x + 3y + 5z &= 7, \\ 4w + \phantom{x+{}} 4y + 8z &= 8, \\ w + x + 2y + 3z &= 5, \\ x + y + z &= 3. \end{aligned}$$

2.3.2. Construct a 3×4 matrix $\mathbf{A}$ and 3×1 columns $\mathbf{b}$ and $\mathbf{c}$ such that $[\mathbf{A}|\mathbf{b}]$ is the augmented matrix for an inconsistent system, but $[\mathbf{A}|\mathbf{c}]$ is the augmented matrix for a consistent system.

2.3.3. If $\mathbf{A}$ is an $m \times n$ matrix with $rank\,(\mathbf{A}) = m$, explain why the system $[\mathbf{A}|\mathbf{b}]$ must be consistent for every right-hand side $\mathbf{b}$.

2.3.4. Consider two consistent systems whose augmented matrices are of the form $[\mathbf{A}|\mathbf{b}]$ and $[\mathbf{A}|\mathbf{c}]$. That is, they differ only on the right-hand side. Is the system associated with $[\mathbf{A} \mid \mathbf{b}+\mathbf{c}]$ also consistent? Explain why.

2.3.5. Is it possible for a parabola whose equation has the form $y = \alpha+\beta x+\gamma x^2$ to pass through the four points $(0,1)$, $(1,3)$, $(2,15)$, and $(3,37)$? Why?

2.3.6. Consider using floating-point arithmetic (without scaling) to solve the following system:

$$\begin{aligned} .835x + .667y &= .168, \\ .333x + .266y &= .067. \end{aligned}$$

(a) Is the system consistent when 5-digit arithmetic is used?
(b) What happens when 6-digit arithmetic is used?

2.3.7. In order to grow a certain crop, it is recommended that each square foot of ground be treated with 10 units of phosphorous, 9 units of potassium, and 19 units of nitrogen. Suppose that there are three brands of fertilizer on the market— say brand $\mathcal{X}$, brand $\mathcal{Y}$, and brand $\mathcal{Z}$. One pound of brand $\mathcal{X}$ contains 2 units of phosphorous, 3 units of potassium, and 5 units of nitrogen. One pound of brand $\mathcal{Y}$ contains 1 unit of phosphorous, 3 units of potassium, and 4 units of nitrogen. One pound of brand $\mathcal{Z}$ contains only 1 unit of phosphorous and 1 unit of nitrogen. Determine whether or not it is possible to meet exactly the recommendation by applying some combination of the three brands of fertilizer.

2.3.8. Suppose that an augmented matrix $[\mathbf{A}|\mathbf{b}]$ is reduced by means of Gaussian elimination to a row echelon form $[\mathbf{E}|\mathbf{c}]$. If a row of the form

$$(\,0 \quad 0 \quad \cdots \quad 0 \quad | \quad \alpha\,), \qquad \alpha \neq 0$$

does not appear in $[\mathbf{E}|\mathbf{c}]$, is it possible that rows of this form could have appeared at earlier stages in the reduction process? Why?

(b) $\begin{pmatrix} 1 & \frac{1}{2} & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and

$\mathbf{A}_{*2} = \frac{1}{2}\mathbf{A}_{*1}, \quad \mathbf{A}_{*4} = 2\mathbf{A}_{*1} - \mathbf{A}_{*3}, \quad \mathbf{A}_{*6} = 2\mathbf{A}_{*1} - 3\mathbf{A}_{*5}, \quad \mathbf{A}_{*7} = \mathbf{A}_{*3} + \mathbf{A}_{*5}$

2.2.2. No.

2.2.3. The same would have to hold in $\mathbf{E_A}$, and there you can see that this means not all columns can be basic. Remember, $rank\,(\mathbf{A}) =$ number of basic columns.

2.2.4. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\mathbf{A}_{*3}$ is almost a combination of $\mathbf{A}_{*1}$ and $\mathbf{A}_{*2}$. In particular, $\mathbf{A}_{*3} \approx -\mathbf{A}_{*1} + 2\mathbf{A}_{*2}$.

2.2.5. $\mathbf{E}_{*1} = 2\mathbf{E}_{*2} - \mathbf{E}_{*3}$ and $\mathbf{E}_{*2} = \frac{1}{2}\mathbf{E}_{*1} + \frac{1}{2}\mathbf{E}_{*3}$

Solutions for exercises in section 2. 3

2.3.1. (a), (b)—There is no need to do any arithmetic for this one because the right-hand side is entirely zero so that you know (0,0,0) is automatically one solution. (d), (f)

2.3.3. It is always true that $rank\,(\mathbf{A}) \leq rank[\mathbf{A}|\mathbf{b}] \leq m$. Since $rank\,(\mathbf{A}) = m$, it follows that $rank[\mathbf{A}|\mathbf{b}] = rank\,(\mathbf{A})$.

2.3.4. Yes—Consistency implies that $\mathbf{b}$ and $\mathbf{c}$ are each combinations of the basic columns in $\mathbf{A}$. If $\mathbf{b} = \sum \beta_i \mathbf{A}_{*b_i}$ and $\mathbf{c} = \sum \gamma_i \mathbf{A}_{*b_i}$ where the $\mathbf{A}_{*b_i}$'s are the basic columns, then $\mathbf{b} + \mathbf{c} = \sum (\beta_i + \gamma_i)\mathbf{A}_{*b_i} = \sum \xi_i \mathbf{A}_{*b_i}$, where $\xi_i = \beta_i + \gamma_i$ so that $\mathbf{b} + \mathbf{c}$ is also a combination of the basic columns in $\mathbf{A}$.

2.3.5. Yes—because the 4×3 system $\alpha + \beta x_i + \gamma x_i^2 = y_i$ obtained by using the four given points (x_i, y_i) is consistent.

2.3.6. The system is inconsistent using 5-digits but consistent when 6-digits are used.

2.3.7. If x, y, and z denote the number of pounds of the respective brands applied, then the following constraints must be met.

$$\begin{aligned} \text{total \# units of phosphorous} &= 2x + \ \ y + z = 10 \\ \text{total \# units of potassium} &= 3x + 3y \qquad\ = 9 \\ \text{total \# units of nitrogen} &= 5x + 4y + z = 19 \end{aligned}$$

Since this is a consistent system, the recommendation can be satisfied exactly. Of course, the solution tells how much of each brand to apply.

2.3.8. No—if one or more such rows were ever present, how could you possibly eliminate all of them with row operations? You could eliminate all but one, but then there is no way to eliminate the last remaining one, and hence it would have to appear in the final form.