2.4 HOMOGENEOUS SYSTEMS

A system of m linear equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0, \end{aligned}$$

in which the right-hand side consists entirely of 0's is said to be a ***homogeneous system.*** If there is at least one nonzero number on the right-hand side, then the system is called ***nonhomogeneous.*** The purpose of this section is to examine some of the elementary aspects concerning homogeneous systems.

Consistency is never an issue when dealing with homogeneous systems because the zero solution $x_1 = x_2 = \cdots = x_n = 0$ is always one solution regardless of the values of the coefficients. Hereafter, the solution consisting of all zeros is referred to as the ***trivial solution.*** The only question is, "Are there solutions other than the trivial solution, and if so, how can we best describe them?" As before, Gaussian elimination provides the answer.

While reducing the augmented matrix $[\mathbf{A}|\mathbf{0}]$ of a homogeneous system to a row echelon form using Gaussian elimination, the zero column on the right-hand side can never be altered by any of the three elementary row operations. That is, any row echelon form derived from $[\mathbf{A}|\mathbf{0}]$ by means of row operations must also have the form $[\mathbf{E}|\mathbf{0}]$. This means that the last column of 0's is just excess baggage that is not necessary to carry along at each step. Just reduce the coefficient matrix $\mathbf{A}$ to a row echelon form $\mathbf{E}$, and remember that the right-hand side is entirely zero when you execute back substitution. The process is best understood by considering a typical example.

In order to examine the solutions of the homogeneous system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 0, \\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 0, \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 0, \end{aligned} \tag{2.4.1}$$

reduce the coefficient matrix to a row echelon form.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -5 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{E}.$$

Therefore, the original homogeneous system is equivalent to the following reduced homogeneous system:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 0, \\ -\,3x_3 - 3x_4 &= 0. \end{aligned} \tag{2.4.2}$$

Since there are four unknowns but only two equations in this reduced system, it is impossible to extract a unique solution for each unknown. The best we can do is to pick two "basic" unknowns—which will be called the ***basic variables*** and solve for these in terms of the other two unknowns—whose values must remain arbitrary or "free," and consequently they will be referred to as the ***free variables.*** Although there are several possibilities for selecting a set of basic variables, the convention is to *always solve for the unknowns corresponding to the pivotal positions*—or, equivalently, the unknowns corresponding to the basic columns. In this example, the pivots (as well as the basic columns) lie in the first and third positions, so the strategy is to apply back substitution to solve the reduced system (2.4.2) for the basic variables x_1 and x_3 in terms of the free variables x_2 and x_4. The second equation in (2.4.2) yields

$$x_3 = -x_4$$

and substitution back into the first equation produces

$$\begin{aligned} x_1 &= -2x_2 - 2x_3 - 3x_4, \\ &= -2x_2 - 2(-x_4) - 3x_4, \\ &= -2x_2 - x_4. \end{aligned}$$

Therefore, all solutions of the original homogeneous system can be described by saying

$$\begin{aligned} x_1 &= -2x_2 - x_4, \\ x_2 &\ \text{ is "free,"} \\ x_3 &= -x_4, \\ x_4 &\ \text{ is "free."} \end{aligned} \tag{2.4.3}$$

As the free variables x_2 and x_4 range over all possible values, the above expressions describe all possible solutions. For example, when x_2 and x_4 assume the values $x_2 = 1$ and $x_4 = -2$, then the particular solution

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 2, \quad x_4 = -2$$

is produced. When $x_2 = \pi$ and $x_4 = \sqrt{2}$, then another particular solution

$$x_1 = -2\pi - \sqrt{2}, \quad x_2 = \pi, \quad x_3 = -\sqrt{2}, \quad x_4 = \sqrt{2}$$

is generated.

Rather than describing the solution set as illustrated in (2.4.3), future developments will make it more convenient to express the solution set by writing

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \tag{2.4.4}$$

with the understanding that x_2 and x_4 are free variables that can range over all possible numbers. This representation will be called the ***general solution*** of the homogeneous system. This expression for the general solution emphasizes that every solution is some combination of the two particular solutions

$$\mathbf{h}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{h}_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

The fact that $\mathbf{h}_1$ and $\mathbf{h}_2$ are each solutions is clear because $\mathbf{h}_1$ is produced when the free variables assume the values $x_2 = 1$ and $x_4 = 0$, whereas the solution $\mathbf{h}_2$ is generated when $x_2 = 0$ and $x_4 = 1$.

Now consider a general homogeneous system $[\mathbf{A}|\mathbf{0}]$ of m linear equations in n unknowns. If the coefficient matrix is such that $rank(\mathbf{A}) = r$, then it should be apparent from the preceding discussion that there will be exactly r basic variables—corresponding to the positions of the basic columns in $\mathbf{A}$—and exactly $n - r$ free variables—corresponding to the positions of the nonbasic columns in $\mathbf{A}$. Reducing $\mathbf{A}$ to a row echelon form using Gaussian elimination and then using back substitution to solve for the basic variables in terms of the free variables produces the ***general solution,*** which has the form

$$\mathbf{x} = x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}, \tag{2.4.5}$$

where $x_{f_1}, x_{f_2}, \ldots, x_{f_{n-r}}$ are the free variables and where $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}$ are $n \times 1$ columns that represent particular solutions of the system. As the free variables x_{f_i} range over all possible values, the general solution generates all possible solutions.

The general solution does not depend on which row echelon form is used in the sense that using back substitution to solve for the basic variables in terms of the nonbasic variables generates a unique set of particular solutions $\{\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}\}$, regardless of which row echelon form is used. Without going into great detail, one can argue that this is true because using back substitution in any row echelon form to solve for the basic variables must produce exactly the same result as that obtained by completely reducing $\mathbf{A}$ to $\mathbf{E}_\mathbf{A}$ and then solving the reduced homogeneous system for the basic variables. Uniqueness of $\mathbf{E}_\mathbf{A}$ guarantees the uniqueness of the $\mathbf{h}_i$'s.

For example, if the coefficient matrix $\mathbf{A}$ associated with the system (2.4.1) is completely reduced by the Gauss–Jordan procedure to $\mathbf{E}_\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{E}_\mathbf{A},$$

then we obtain the following reduced system:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 0, \\ x_3 + x_4 &= 0. \end{aligned}$$

Solving for the basic variables x_1 and x_3 in terms of x_2 and x_4 produces exactly the same result as given in (2.4.3) and hence generates exactly the same general solution as shown in (2.4.4).

Because it avoids the back substitution process, you may find it more convenient to use the Gauss–Jordan procedure to reduce $\mathbf{A}$ completely to $\mathbf{E_A}$ and then construct the general solution directly from the entries in $\mathbf{E_A}$. This approach usually will be adopted in the examples and exercises.

As was previously observed, all homogeneous systems are consistent because the trivial solution consisting of all zeros is always one solution. The natural question is, "When is the trivial solution the *only* solution?" In other words, we wish to know when a homogeneous system possesses a unique solution. The form of the general solution (2.4.5) makes the answer transparent. As long as there is at least one free variable, then it is clear from (2.4.5) that there will be an infinite number of solutions. Consequently, the trivial solution is the only solution if and only if there are no free variables. Because the number of free variables is given by $n - r$, where $r = rank(\mathbf{A})$, the previous statement can be reformulated to say that *a homogeneous system possesses a unique solution—the trivial solution—if and only if* $rank(\mathbf{A}) = n$.

Example 2.4.1

The homogeneous system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0, \\ 2x_1 + 5x_2 + 7x_3 &= 0, \\ 3x_1 + 6x_2 + 8x_3 &= 0, \end{aligned}$$

has only the trivial solution because

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 3 & 6 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} = \mathbf{E}$$

shows that $rank(\mathbf{A}) = n = 3$. Indeed, it is also obvious from $\mathbf{E}$ that applying back substitution in the system $[\mathbf{E}|\mathbf{0}]$ yields only the trivial solution.

Example 2.4.2

Problem: Explain why the following homogeneous system has infinitely many solutions, and exhibit the general solution:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0, \\ 2x_1 + 5x_2 + 7x_3 &= 0, \\ 3x_1 + 6x_2 + 6x_3 &= 0. \end{aligned}$$

Solution:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 3 & 6 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{E}$$

shows that $rank(\mathbf{A}) = 2 < n = 3$. Since the basic columns lie in positions one and two, x_1 and x_2 are the basic variables while x_3 is free. Using back substitution on $[\mathbf{E}|\mathbf{0}]$ to solve for the basic variables in terms of the free variable produces $x_2 = -3x_3$ and $x_1 = -2x_2 - 2x_3 = 4x_3$, so the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}, \quad \text{where} \quad x_3 \text{ is free.}$$

That is, every solution is a multiple of the one particular solution $\mathbf{h}_1 = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$.

Summary

Let $\mathbf{A}_{m\times n}$ be the coefficient matrix for a homogeneous system of m linear equations in n unknowns, and suppose $rank(\mathbf{A}) = r$.

- The unknowns that correspond to the positions of the basic columns (i.e., the pivotal positions) are called the ***basic variables***, and the unknowns corresponding to the positions of the nonbasic columns are called the ***free variables.***
- There are exactly r basic variables and $n - r$ free variables.
- To describe all solutions, reduce $\mathbf{A}$ to a row echelon form using Gaussian elimination, and then use back substitution to solve for the basic variables in terms of the free variables. This produces the ***general solution*** that has the form

$$\mathbf{x} = x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r},$$

  where the terms $x_{f_1}, x_{f_2}, \ldots, x_{f_{n-r}}$ are the free variables and where $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}$ are $n \times 1$ columns that represent particular solutions of the homogeneous system. The $\mathbf{h}_i$'s are independent of which row echelon form is used in the back substitution process. As the free variables x_{f_i} range over all possible values, the general solution generates all possible solutions.
- A homogeneous system possesses a unique solution (the trivial solution) if and only if $rank(\mathbf{A}) = n$—i.e., if and only if there are no free variables.

Exercises for section 2.4

2.4.1. Determine the general solution for each of the following homogeneous systems.

$$\text{(a)}\quad \begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 &= 0,\\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 0,\\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 0. \end{aligned} \qquad \text{(b)}\quad \begin{aligned} 2x + y + z &= 0,\\ 4x + 2y + z &= 0,\\ 6x + 3y + z &= 0,\\ 8x + 4y + z &= 0. \end{aligned}$$

$$\text{(c)}\quad \begin{aligned} x_1 + x_2 + 2x_3 \phantom{{}+3x_4} &= 0,\\ 3x_1 \phantom{{}+x_2} + 3x_3 + 3x_4 &= 0,\\ 2x_1 + x_2 + 3x_3 + x_4 &= 0,\\ x_1 + 2x_2 + 3x_3 - x_4 &= 0. \end{aligned} \qquad \text{(d)}\quad \begin{aligned} 2x + y + z &= 0,\\ 4x + 2y + z &= 0,\\ 6x + 3y + z &= 0,\\ 8x + 5y + z &= 0. \end{aligned}$$

2.4.2. Among all solutions that satisfy the homogeneous system

$$\begin{aligned} x + 2y + z &= 0,\\ 2x + 4y + z &= 0,\\ x + 2y - z &= 0, \end{aligned}$$

determine those that also satisfy the nonlinear constraint $y - xy = 2z$.

2.4.3. Consider a homogeneous system whose coefficient matrix is

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 & 1\\ 2 & 4 & -1 & 3 & 8\\ 1 & 2 & 3 & 5 & 7\\ 2 & 4 & 2 & 6 & 2\\ 3 & 6 & 1 & 7 & -3 \end{pmatrix}.$$

First transform $\mathbf{A}$ to an unreduced row echelon form to determine the general solution of the associated homogeneous system. Then reduce $\mathbf{A}$ to $\mathbf{E_A}$, and show that the same general solution is produced.

2.4.4. If $\mathbf{A}$ is the coefficient matrix for a homogeneous system consisting of four equations in eight unknowns and if there are five free variables, what is $rank\,(\mathbf{A})$?

2.4.5. Suppose that $\mathbf{A}$ is the coefficient matrix for a homogeneous system of four equations in six unknowns and suppose that $\mathbf{A}$ has at least one nonzero row.

(a) Determine the fewest number of free variables that are possible.

(b) Determine the maximum number of free variables that are possible.

2.4.6. Explain why a homogeneous system of m equations in n unknowns where $m < n$ must always possess an infinite number of solutions.

2.4.7. Construct a homogeneous system of three equations in four unknowns that has

$$x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

as its general solution.

2.4.8. If $\mathbf{c}_1$ and $\mathbf{c}_2$ are columns that represent two particular solutions of the same homogeneous system, explain why the sum $\mathbf{c}_1 + \mathbf{c}_2$ must also represent a solution of this system.

Solutions for exercises in section 2. 4

2.4.1. (a) $x_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-1\\0\\-1\\1\end{pmatrix}$ (b) $y\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$ (c) $x_3\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}+x_4\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$

(d) The trivial solution is the only solution.

2.4.2. $\begin{pmatrix}0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-\frac{1}{2}\\0\end{pmatrix}$

2.4.3. $x_2\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-2\\0\\-1\\1\\0\end{pmatrix}$

2.4.4. $rank\,(\mathbf{A})=3$

2.4.5. (a) 2—because the maximum rank is 4. (b) 5—because the minimum rank is 1.

2.4.6. Because $r=rank\,(\mathbf{A})\le m<n \implies n-r>0$.

2.4.7. There are many different correct answers. One approach is to answer the question "What must $\mathbf{E_A}$ look like?" The form of the general solution tells you that $rank\,(\mathbf{A})=2$ and that the first and third columns are basic. Consequently, $\mathbf{E_A}=\begin{pmatrix}1&\alpha&0&\beta\\0&0&1&\gamma\\0&0&0&0\end{pmatrix}$ so that $x_1=-\alpha x_2-\beta x_4$ and $x_3=-\gamma x_4$ gives rise to the general solution $x_2\begin{pmatrix}-\alpha\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-\beta\\0\\-\gamma\\1\end{pmatrix}$. Therefore, $\alpha=2$, $\beta=3$, and $\gamma=-2$. Any matrix $\mathbf{A}$ obtained by performing row operations to $\mathbf{E_A}$ will be the coefficient matrix for a homogeneous system with the desired general solution.

2.4.8. If $\sum_i x_{f_i}\mathbf{h}_i$ is the general solution, then there must exist scalars α_i and β_i such that $\mathbf{c}_1=\sum_i\alpha_i\mathbf{h}_i$ and $\mathbf{c}_2=\sum_i\beta_i\mathbf{h}_i$. Therefore, $\mathbf{c}_1+\mathbf{c}_2=\sum_i(\alpha_i+\beta_i)\mathbf{h}_i$, and this shows that $\mathbf{c}_1+\mathbf{c}_2$ is the solution obtained when the free variables x_{f_i} assume the values $x_{f_i}=\alpha_i+\beta_i$.

Solutions for exercises in section 2. 5

2.5.1. (a) $\begin{pmatrix}1\\0\\2\\0\end{pmatrix}+x_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-1\\0\\-1\\1\end{pmatrix}$ (b) $\begin{pmatrix}1\\0\\2\end{pmatrix}+y\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$