2.5 NONHOMOGENEOUS SYSTEMS

Recall that a system of m linear equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

is said to be ***nonhomogeneous*** whenever $b_i \neq 0$ for at least one i. Unlike homogeneous systems, a nonhomogeneous system may be inconsistent and the techniques of §2.3 must be applied in order to determine if solutions do indeed exist. Unless otherwise stated, it is assumed that all systems in this section are consistent.

To describe the set of all possible solutions of a consistent nonhomogeneous system, construct a general solution by exactly the same method used for homogeneous systems as follows.

- Use Gaussian elimination to reduce the associated augmented matrix $[\mathbf{A}|\mathbf{b}]$ to a row echelon form $[\mathbf{E}|\mathbf{c}]$.

- Identify the basic variables and the free variables in the same manner described in §2.4.

- Apply back substitution to $[\mathbf{E}|\mathbf{c}]$ and solve for the basic variables in terms of the free variables.

- Write the result in the form

$$\mathbf{x} = \mathbf{p} + x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}, \tag{2.5.1}$$

  where $x_{f_1}, x_{f_2}, \ldots, x_{f_{n-r}}$ are the free variables and $\mathbf{p}, \mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}$ are $n \times 1$ columns. This is the ***general solution*** of the nonhomogeneous system.

As the free variables x_{f_i} range over all possible values, the general solution (2.5.1) generates all possible solutions of the system $[\mathbf{A}|\mathbf{b}]$. Just as in the homogeneous case, the columns $\mathbf{h_i}$ and $\mathbf{p}$ are independent of which row echelon form $[\mathbf{E}|\mathbf{c}]$ is used. Therefore, $[\mathbf{A}|\mathbf{b}]$ may be completely reduced to $\mathbf{E}_{[\mathbf{A}|\mathbf{b}]}$ by using the Gauss–Jordan method thereby avoiding the need to perform back substitution. We will use this approach whenever it is convenient.

The difference between the general solution of a nonhomogeneous system and the general solution of a homogeneous system is the column $\mathbf{p}$ that appears in (2.5.1). To understand why $\mathbf{p}$ appears and where it comes from, consider the nonhomogeneous system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 4, \\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 5, \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 7, \end{aligned} \tag{2.5.2}$$

in which the coefficient matrix is the same as the coefficient matrix for the homogeneous system (2.4.1) used in the previous section. If $[\mathbf{A}|\mathbf{b}]$ is completely reduced by the Gauss–Jordan procedure to $\mathbf{E}_{[\mathbf{A}|\mathbf{b}]}$

$$[\mathbf{A}|\mathbf{b}] = \left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 & 5 \\ 3 & 6 & 1 & 4 & 7 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \mathbf{E}_{[\mathbf{A}|\mathbf{b}]},$$

then the following reduced system is obtained:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 2, \\ x_3 + x_4 &= 1. \end{aligned}$$

Solving for the basic variables, x_1 and x_3, in terms of the free variables, x_2 and x_4, produces

$$\begin{aligned} x_1 &= 2 - 2x_2 - x_4, \\ x_2 &\text{ is "free,"} \\ x_3 &= 1 - x_4, \\ x_4 &\text{ is "free."} \end{aligned}$$

The general solution is obtained by writing these statements in the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 - 2x_2 - x_4 \\ x_2 \\ 1 - x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}. \tag{2.5.3}$$

As the free variables x_2 and x_4 range over all possible numbers, this generates all possible solutions of the nonhomogeneous system (2.5.2). Notice that the column $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ in (2.5.3) is a *particular solution* of the nonhomogeneous system (2.5.2)—it is the solution produced when the free variables assume the values $x_2 = 0$ and $x_4 = 0$.

Furthermore, recall from (2.4.4) that the general solution of the associated homogeneous system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 0,\\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 0,\\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 0, \end{aligned} \tag{2.5.4}$$

is given by

$$\begin{pmatrix} -2x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

That is, the general solution of the associated homogeneous system (2.5.4) is a part of the general solution of the original nonhomogeneous system (2.5.2).

These two observations can be combined by saying that *the general solution of the nonhomogeneous system is given by a particular solution plus the general solution of the associated homogeneous system.*[14]

To see that the previous statement is always true, suppose $[\mathbf{A}|\mathbf{b}]$ represents a general $m \times n$ consistent system where $rank\,(\mathbf{A}) = r$. Consistency guarantees that $\mathbf{b}$ is a nonbasic column in $[\mathbf{A}|\mathbf{b}]$, and hence the basic columns in $[\mathbf{A}|\mathbf{b}]$ are in the same positions as the basic columns in $[\mathbf{A}|\mathbf{0}]$ so that the nonhomogeneous system and the associated homogeneous system have exactly the same set of basic variables as well as free variables. Furthermore, it is not difficult to see that

$$\mathbf{E}_{[\mathbf{A}|\mathbf{0}]} = [\mathbf{E}_\mathbf{A}|\mathbf{0}] \quad \text{and} \quad \mathbf{E}_{[\mathbf{A}|\mathbf{b}]} = [\mathbf{E}_\mathbf{A}|\mathbf{c}],$$

where $\mathbf{c}$ is some column of the form $\mathbf{c} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. This means that if you solve the i^{th} equation in the reduced homogeneous system for the i^{th} basic variable x_{b_i} in terms of the free variables $x_{f_i}, x_{f_{i+1}}, \ldots, x_{f_{n-r}}$ to produce

$$x_{b_i} = \alpha_i x_{f_i} + \alpha_{i+1} x_{f_{i+1}} + \cdots + \alpha_{n-r} x_{f_{n-r}}, \tag{2.5.5}$$

then the solution for the i^{th} basic variable in the reduced nonhomogeneous system must have the form

$$x_{b_i} = \xi_i + \alpha_i x_{f_i} + \alpha_{i+1} x_{f_{i+1}} + \cdots + \alpha_{n-r} x_{f_{n-r}}. \tag{2.5.6}$$

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[14] For those students who have studied differential equations, this statement should have a familiar ring. Exactly the same situation holds for the general solution to a linear differential equation. This is no accident—it is due to the inherent linearity in both problems. More will be said about this issue later in the text.

That is, the two solutions differ only in the fact that the latter contains the constant ξ_i. Consider organizing the expressions (2.5.5) and (2.5.6) so as to construct the respective general solutions. If the general solution of the homogeneous system has the form

$$\mathbf{x} = x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r},$$

then it is apparent that the general solution of the nonhomogeneous system must have a similar form

$$\mathbf{x} = \mathbf{p} + x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r} \tag{2.5.7}$$

in which the column $\mathbf{p}$ contains the constants ξ_i along with some 0's—the ξ_i 's occupy positions in $\mathbf{p}$ that correspond to the positions of the basic columns, and 0's occupy all other positions. The column $\mathbf{p}$ represents one particular solution to the nonhomogeneous system because it is the solution produced when the free variables assume the values $x_{f_1} = x_{f_2} = \cdots = x_{f_{n-r}} = 0$.

Example 2.5.1

Problem: Determine the general solution of the following nonhomogeneous system and compare it with the general solution of the associated homogeneous system:

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 1,\\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 3x_5 &= 1,\\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 2x_5 &= 2,\\ 3x_1 + 5x_2 + 8x_3 + 6x_4 + 5x_5 &= 3. \end{aligned}$$

Solution: Reducing the augmented matrix $[\mathbf{A}|\mathbf{b}]$ to $\mathbf{E}_{[\mathbf{A}|\mathbf{b}]}$ yields

$$\mathbf{A} = \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1\\ 2 & 2 & 4 & 4 & 3 & 1\\ 2 & 2 & 4 & 4 & 2 & 2\\ 3 & 5 & 8 & 6 & 5 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1\\ 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 2 & 2 & 0 & 2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1\\ 0 & 2 & 2 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 2 & 2 & 1 & 1\\ 0 & 1 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & 1\\ 0 & 1 & 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & 1\\ 0 & 1 & 1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = \mathbf{E}_{[\mathbf{A}|\mathbf{b}]}.$$

Observe that the system is indeed consistent because the last column is nonbasic. Solve the reduced system for the basic variables x_1, x_2, and x_5 in terms of the free variables x_3 and x_4 to obtain

$$\begin{aligned} x_1 &= 1 - x_3 - 2x_4, \\ x_2 &= 1 - x_3, \\ x_3 &\text{ is "free,"} \\ x_4 &\text{ is "free,"} \\ x_5 &= -1. \end{aligned}$$

The general solution to the nonhomogeneous system is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 - x_3 - 2x_4 \\ 1 - x_3 \\ x_3 \\ x_4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The general solution of the associated homogeneous system is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_3 - 2x_4 \\ -x_3 \\ x_3 \\ x_4 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

You should verify for yourself that

$$\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

is indeed a particular solution to the nonhomogeneous system and that

$$\mathbf{h}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{h}_4 = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

are particular solutions to the associated homogeneous system.

Now turn to the question, "When does a consistent system have a unique solution?" It is known from (2.5.7) that the general solution of a consistent $m \times n$ nonhomogeneous system $[\mathbf{A}|\mathbf{b}]$ with $rank(\mathbf{A}) = r$ is given by

$$\mathbf{x} = \mathbf{p} + x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r},$$

where

$$x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}$$

is the general solution of the associated homogeneous system. Consequently, it is evident that the nonhomogeneous system $[\mathbf{A}|\mathbf{b}]$ will have a unique solution (namely, $\mathbf{p}$) if and only if there are no free variables—i.e., if and only if $r = n$ (= number of unknowns)—this is equivalent to saying that the associated homogeneous system $[\mathbf{A}|\mathbf{0}]$ has only the trivial solution.

Example 2.5.2

Consider the following nonhomogeneous system:

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 2, \\ x_1 + 2x_2 + 3x_3 &= 1, \\ x_1 \qquad\quad + x_3 &= -3, \\ 2x_1 + 4x_2 \qquad\quad &= 8. \end{aligned}$$

Reducing $[\mathbf{A}|\mathbf{b}]$ to $\mathbf{E}_{[\mathbf{A}|\mathbf{b}]}$ yields

$$[\mathbf{A}|\mathbf{b}] = \left(\begin{array}{ccc|c} 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & -3 \\ 2 & 4 & 0 & 8 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) = \mathbf{E}_{[\mathbf{A}|\mathbf{b}]}.$$

The system is consistent because the last column is nonbasic. There are several ways to see that the system has a unique solution. Notice that

$$rank(\mathbf{A}) = 3 = \text{ number of unknowns},$$

which is the same as observing that there are no free variables. Furthermore, the associated homogeneous system clearly has only the trivial solution. Finally, because we completely reduced $[\mathbf{A}|\mathbf{b}]$ to $\mathbf{E}_{[\mathbf{A}|\mathbf{b}]}$, it is obvious that there is only one solution possible and that it is given by $\mathbf{p} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$.

Summary

Let $[\mathbf{A}|\mathbf{b}]$ be the augmented matrix for a consistent $m \times n$ nonhomogeneous system in which $rank\,(\mathbf{A}) = r$.

- Reducing $[\mathbf{A}|\mathbf{b}]$ to a row echelon form using Gaussian elimination and then solving for the basic variables in terms of the free variables leads to the ***general solution***

$$\mathbf{x} = \mathbf{p} + x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}.$$

  As the free variables x_{f_i} range over all possible values, this general solution generates all possible solutions of the system.

- Column $\mathbf{p}$ is a particular solution of the nonhomogeneous system.
- The expression $x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}$ is the general solution of the associated homogeneous system.
- Column $\mathbf{p}$ as well as the columns $\mathbf{h}_i$ are independent of the row echelon form to which $[\mathbf{A}|\mathbf{b}]$ is reduced.
- The system possesses a unique solution if and only if any of the following is true.
  - ▷ $rank\,(\mathbf{A}) = n =$ number of unknowns.
  - ▷ There are no free variables.
  - ▷ The associated homogeneous system possesses only the trivial solution.

Exercises for section 2.5

2.5.1. Determine the general solution for each of the following nonhomogeneous systems.

(a)
$$\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 &= 3, \\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 4, \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 5. \end{aligned}$$

(b)
$$\begin{aligned} 2x + y + z &= 4, \\ 4x + 2y + z &= 6, \\ 6x + 3y + z &= 8, \\ 8x + 4y + z &= 10. \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1, \\ 3x_1 + 3x_3 + 3x_4 &= 6, \\ 2x_1 + x_2 + 3x_3 + x_4 &= 3, \\ x_1 + 2x_2 + 3x_3 - x_4 &= 0. \end{aligned}$$

(d)
$$\begin{aligned} 2x + y + z &= 2, \\ 4x + 2y + z &= 5, \\ 6x + 3y + z &= 8, \\ 8x + 5y + z &= 8. \end{aligned}$$

2.5.2. Among the solutions that satisfy the set of linear equations

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 1,\\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 3x_5 &= 1,\\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 2x_5 &= 2,\\ 3x_1 + 5x_2 + 8x_3 + 6x_4 + 5x_5 &= 3, \end{aligned}$$

find all those that also satisfy the following two constraints:

$$\begin{aligned} (x_1 - x_2)^2 - 4x_5^2 &= 0,\\ x_3^2 - x_5^2 &= 0. \end{aligned}$$

2.5.3. In order to grow a certain crop, it is recommended that each square foot of ground be treated with 10 units of phosphorous, 9 units of potassium, and 19 units of nitrogen. Suppose that there are three brands of fertilizer on the market—say brand $\mathcal{X}$, brand $\mathcal{Y}$, and brand $\mathcal{Z}$. One pound of brand $\mathcal{X}$ contains 2 units of phosphorous, 3 units of potassium, and 5 units of nitrogen. One pound of brand $\mathcal{Y}$ contains 1 unit of phosphorous, 3 units of potassium, and 4 units of nitrogen. One pound of brand $\mathcal{Z}$ contains only 1 unit of phosphorous and 1 unit of nitrogen.

(a) Take into account the obvious fact that a negative number of pounds of any brand can never be applied, and suppose that because of the way fertilizer is sold only an integral number of pounds of each brand will be applied. Under these constraints, determine all possible combinations of the three brands that can be applied to satisfy the recommendations exactly.

(b) Suppose that brand $\mathcal{X}$ costs \$1 per pound, brand $\mathcal{Y}$ costs \$6 per pound, and brand $\mathcal{Z}$ costs \$3 per pound. Determine the least expensive solution that will satisfy the recommendations exactly as well as the constraints of part (a).

2.5.4. Consider the following system:

$$\begin{aligned} 2x + 2y + 3z &= 0,\\ 4x + 8y + 12z &= -4,\\ 6x + 2y + \alpha z &= 4. \end{aligned}$$

(a) Determine all values of α for which the system is consistent.

(b) Determine all values of α for which there is a unique solution, and compute the solution for these cases.

(c) Determine all values of α for which there are infinitely many different solutions, and give the general solution for these cases.

2.5.5. If columns $\mathbf{s}_1$ and $\mathbf{s}_2$ are particular solutions of the same nonhomogeneous system, must it be the case that the sum $\mathbf{s}_1 + \mathbf{s}_2$ is also a solution?

2.5.6. Suppose that $[\mathbf{A}|\mathbf{b}]$ is the augmented matrix for a consistent system of m equations in n unknowns where $m \geq n$. What must $\mathbf{E_A}$ look like when the system possesses a unique solution?

2.5.7. Construct a nonhomogeneous system of three equations in four unknowns that has

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

as its general solution.

2.5.8. Consider using floating-point arithmetic (without partial pivoting or scaling) to solve the system represented by the following augmented matrix:

$$\left(\begin{array}{ccc|c} .835 & .667 & .5 & .168 \\ .333 & .266 & .1994 & .067 \\ 1.67 & 1.334 & 1.1 & .436 \end{array}\right).$$

(a) Determine the 4-digit general solution.
(b) Determine the 5-digit general solution.
(c) Determine the 6-digit general solution.

Solutions for exercises in section 2. 4

2.4.1. (a) $x_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-1\\0\\-1\\1\end{pmatrix}$ (b) $y\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$ (c) $x_3\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}+x_4\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$

(d) The trivial solution is the only solution.

2.4.2. $\begin{pmatrix}0\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-\frac{1}{2}\\0\end{pmatrix}$

2.4.3. $x_2\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-2\\0\\-1\\1\\0\end{pmatrix}$

2.4.4. $rank\,(\mathbf{A})=3$

2.4.5. (a) 2—because the maximum rank is 4. (b) 5—because the minimum rank is 1.

2.4.6. Because $r=rank\,(\mathbf{A})\le m<n \implies n-r>0$.

2.4.7. There are many different correct answers. One approach is to answer the question "What must $\mathbf{E_A}$ look like?" The form of the general solution tells you that $rank\,(\mathbf{A})=2$ and that the first and third columns are basic. Consequently, $\mathbf{E_A}=\begin{pmatrix}1&\alpha&0&\beta\\0&0&1&\gamma\\0&0&0&0\end{pmatrix}$ so that $x_1=-\alpha x_2-\beta x_4$ and $x_3=-\gamma x_4$ gives rise to the general solution $x_2\begin{pmatrix}-\alpha\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-\beta\\0\\-\gamma\\1\end{pmatrix}$. Therefore, $\alpha=2$, $\beta=3$, and $\gamma=-2$. Any matrix $\mathbf{A}$ obtained by performing row operations to $\mathbf{E_A}$ will be the coefficient matrix for a homogeneous system with the desired general solution.

2.4.8. If $\sum_i x_{f_i}\mathbf{h}_i$ is the general solution, then there must exist scalars α_i and β_i such that $\mathbf{c}_1=\sum_i \alpha_i\mathbf{h}_i$ and $\mathbf{c}_2=\sum_i \beta_i\mathbf{h}_i$. Therefore, $\mathbf{c}_1+\mathbf{c}_2=\sum_i(\alpha_i+\beta_i)\mathbf{h}_i$, and this shows that $\mathbf{c}_1+\mathbf{c}_2$ is the solution obtained when the free variables x_{f_i} assume the values $x_{f_i}=\alpha_i+\beta_i$.

Solutions for exercises in section 2. 5

2.5.1. (a) $\begin{pmatrix}1\\0\\2\\0\end{pmatrix}+x_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-1\\0\\-1\\1\end{pmatrix}$ (b) $\begin{pmatrix}1\\0\\2\end{pmatrix}+y\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix}$

2.5.2. From Example 2.5.1, the solutions of the linear equations are:

$$\begin{aligned} x_1 &= 1 - x_3 - 2x_4 \\ x_2 &= 1 - x_3 \\ x_3 &\text{ is free} \\ x_4 &\text{ is free} \\ x_5 &= -1 \end{aligned}$$

Substitute these into the two constraints to get $x_3 = \pm 1$ and $x_4 = \pm 1$. Thus there are exactly four solutions:

$$\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ -1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

2.5.3. (a) $\{(3,0,4),\ (2,1,5),\ (1,2,6),\ (0,3,7)\}$ See the solution to Exercise 2.3.7 for the underlying system. (b) $(3,0,4)$ costs \$15 and is least expensive.

2.5.4. (a) Consistent for all α. (b) $\alpha \neq 3$, in which case the solution is $(1,-1,0)$.

(c) $\alpha = 3$, in which case the general solution is $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ -\frac{3}{2} \\ 1 \end{pmatrix}$.

2.5.5. No

2.5.6.

$$\mathbf{E_A} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m \times n}$$

2.5.7. See the solution to Exercise 2.4.7.

2.5.8. (a) $\begin{pmatrix} -.3976 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} -.7988 \\ 1 \\ 0 \end{pmatrix}$ (b) There are no solutions in this case.

(c) $\begin{pmatrix} 1.43964 \\ -2.3 \\ 1 \end{pmatrix}$