3.2 ADDITION AND TRANSPOSITION

In the previous chapters, matrix language and notation were used simply to formulate some of the elementary concepts surrounding linear systems. The purpose now is to turn this language into a mathematical theory.[16]

Unless otherwise stated, a ***scalar*** is a complex number. Real numbers are a subset of the complex numbers, and hence real numbers are also scalar quantities. In the early stages, there is little harm in thinking only in terms of real scalars. Later on, however, the necessity for dealing with complex numbers will be unavoidable. Throughout the text, $\Re$ will denote the set of real numbers, and $\mathcal{C}$ will denote the complex numbers. The set of all n-tuples of real numbers will be denoted by $\Re^n$, and the set of all complex n-tuples will be denoted by $\mathcal{C}^n$. For example, $\Re^2$ is the set of all ordered pairs of real numbers (i.e., the standard cartesian plane), and $\Re^3$ is ordinary 3-space. Analogously, $\Re^{m\times n}$ and $\mathcal{C}^{m\times n}$ denote the $m \times n$ matrices containing real numbers and complex numbers, respectively.

Matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are defined to be ***equal matrices*** when $\mathbf{A}$ and $\mathbf{B}$ have the same shape and corresponding entries are equal. That is, $a_{ij} = b_{ij}$ for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. In particular, this definition applies to arrays such as $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$. Even though $\mathbf{u}$ and $\mathbf{v}$ describe exactly the same point in 3-space, we cannot consider them to be equal matrices because they have different shapes. An array (or matrix) consisting of a single column, such as $\mathbf{u}$, is called a ***column vector***, while an array consisting of a single row, such as $\mathbf{v}$, is called a ***row vector***.

Addition of Matrices

If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices, the ***sum*** of $\mathbf{A}$ and $\mathbf{B}$ is defined to be the $m \times n$ matrix $\mathbf{A}+\mathbf{B}$ obtained by adding corresponding entries. That is,

$$[\mathbf{A}+\mathbf{B}]_{ij} = [\mathbf{A}]_{ij} + [\mathbf{B}]_{ij} \quad \text{for each } i \text{ and } j.$$

For example,

$$\begin{pmatrix} -2 & x & 3 \\ z+3 & 4 & -y \end{pmatrix} + \begin{pmatrix} 2 & 1-x & -2 \\ -3 & 4+x & 4+y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ z & 8+x & 4 \end{pmatrix}.$$

[16] The great French mathematician Pierre-Simon Laplace (1749–1827) said that, "Such is the advantage of a well-constructed language that its simplified notation often becomes the source of profound theories." The theory of matrices is a testament to the validity of Laplace's statement.

The symbol "+" is used two different ways—it denotes addition between scalars in some places and addition between matrices at other places. Although these are two distinct algebraic operations, no ambiguities will arise if the context in which "+" appears is observed. Also note that the requirement that $\mathbf{A}$ and $\mathbf{B}$ have the same shape prevents adding a row to a column, even though the two may contain the same number of entries.

The matrix $(-\mathbf{A})$, called the ***additive inverse*** of $\mathbf{A}$, is defined to be the matrix obtained by negating each entry of $\mathbf{A}$. That is, if $\mathbf{A} = [a_{ij}]$, then $-\mathbf{A} = [-a_{ij}]$. This allows matrix subtraction to be defined in the natural way. For two matrices of the same shape, the ***difference*** $\mathbf{A} - \mathbf{B}$ is defined to be the matrix $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$ so that

$$[\mathbf{A} - \mathbf{B}]_{ij} = [\mathbf{A}]_{ij} - [\mathbf{B}]_{ij} \quad \text{for each } i \text{ and } j.$$

Since matrix addition is defined in terms of scalar addition, the familiar algebraic properties of scalar addition are inherited by matrix addition as detailed below.

Properties of Matrix Addition

For $m \times n$ matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, the following properties hold.

- Closure property: $\mathbf{A} + \mathbf{B}$ is again an $m \times n$ matrix.
- Associative property: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.
- Commutative property: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.
- Additive identity: The $m \times n$ matrix $\mathbf{0}$ consisting of all zeros has the property that $\mathbf{A} + \mathbf{0} = \mathbf{A}$.
- Additive inverse: The $m \times n$ matrix $(-\mathbf{A})$ has the property that $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

Another simple operation that is derived from scalar arithmetic is as follows.

Scalar Multiplication

The product of a scalar α times a matrix $\mathbf{A}$, denoted by $\alpha\mathbf{A}$, is defined to be the matrix obtained by multiplying each entry of $\mathbf{A}$ by α. That is, $[\alpha\mathbf{A}]_{ij} = \alpha[\mathbf{A}]_{ij}$ for each i and j.

For example,

$$2\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 4 \\ 2 & 8 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 4 \\ 6 & 8 \\ 0 & 2 \end{pmatrix}.$$

The rules for combining addition and scalar multiplication are what you might suspect they should be. Some of the important ones are listed below.

Properties of Scalar Multiplication

For $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ and for scalars α and β, the following properties hold.

Closure property:	$\alpha\mathbf{A}$ is again an $m \times n$ matrix.
Associative property:	$(\alpha\beta)\mathbf{A} = \alpha(\beta\mathbf{A})$.
Distributive property:	$\alpha(\mathbf{A}+\mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$. Scalar multiplication is distributed over matrix addition.
Distributive property:	$(\alpha+\beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$. Scalar multiplication is distributed over scalar addition.
Identity property:	$1\mathbf{A} = \mathbf{A}$. The number 1 is an identity element under scalar multiplication.

Other properties such as $\alpha\mathbf{A} = \mathbf{A}\alpha$ could have been listed, but the properties singled out pave the way for the definition of a vector space on p. 160.

A matrix operation that's not derived from scalar arithmetic is *transposition* as defined below.

Transpose

The ***transpose*** of $\mathbf{A}_{m\times n}$ is defined to be the $n \times m$ matrix $\mathbf{A}^T$ obtained by interchanging rows and columns in $\mathbf{A}$. More precisely, if $\mathbf{A} = [a_{ij}]$, then $[\mathbf{A}^T]_{ij} = a_{ji}$. For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}.$$

It should be evident that for all matrices, $\left(\mathbf{A}^T\right)^T = \mathbf{A}$.

Whenever a matrix contains complex entries, the operation of complex conjugation almost always accompanies the transpose operation. (Recall that the complex conjugate of $z = a + \mathrm{i}b$ is defined to be $\overline{z} = a - \mathrm{i}b$.)

Conjugate Transpose

For $\mathbf{A} = [a_{ij}]$, the ***conjugate matrix*** is defined to be $\overline{\mathbf{A}} = [\overline{a}_{ij}]$, and the ***conjugate transpose*** of $\mathbf{A}$ is defined to be $\bar{\mathbf{A}}^T = \overline{\mathbf{A}^T}$. From now on, $\bar{\mathbf{A}}^T$ will be denoted by $\mathbf{A}^*$, so $[\mathbf{A}^*]_{ij} = \overline{a}_{ji}$. For example,

$$\begin{pmatrix} 1-4\mathrm{i} & \mathrm{i} & 2 \\ 3 & 2+\mathrm{i} & 0 \end{pmatrix}^* = \begin{pmatrix} 1+4\mathrm{i} & 3 \\ -\mathrm{i} & 2-\mathrm{i} \\ 2 & 0 \end{pmatrix}.$$

$(\mathbf{A}^*)^* = \mathbf{A}$ for all matrices, and $\mathbf{A}^* = \mathbf{A}^T$ whenever $\mathbf{A}$ contains only real entries. Sometimes the matrix $\mathbf{A}^*$ is called the ***adjoint*** of $\mathbf{A}$.

The transpose (and conjugate transpose) operation is easily combined with matrix addition and scalar multiplication. The basic rules are given below.

Properties of the Transpose

If $\mathbf{A}$ and $\mathbf{B}$ are two matrices of the same shape, and if α is a scalar, then each of the following statements is true.

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \quad \text{and} \quad (\mathbf{A}+\mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*. \tag{3.2.1}$$

$$(\alpha\mathbf{A})^T = \alpha\mathbf{A}^T \quad \text{and} \quad (\alpha\mathbf{A})^* = \overline{\alpha}\mathbf{A}^*. \tag{3.2.2}$$

Proof.[17] We will prove that (3.2.1) and (3.2.2) hold for the transpose operation. The proofs of the statements involving conjugate transposes are similar and are left as exercises. For each i and j, it is true that

$$[(\mathbf{A}+\mathbf{B})^T]_{ij} = [\mathbf{A}+\mathbf{B}]_{ji} = [\mathbf{A}]_{ji} + [\mathbf{B}]_{ji} = [\mathbf{A}^T]_{ij} + [\mathbf{B}^T]_{ij} = [\mathbf{A}^T + \mathbf{B}^T]_{ij}.$$

---

[17] Computers can outperform people in many respects in that they do arithmetic much faster and more accurately than we can, and they are now rather adept at symbolic computation and mechanical manipulation of formulas. But computers can't do mathematics—people still hold the monopoly. Mathematics emanates from the uniquely human capacity to reason abstractly in a creative and logical manner, and learning mathematics goes hand-in-hand with learning how to reason abstractly and create logical arguments. This is true regardless of whether your orientation is applied or theoretical. For this reason, formal proofs will appear more frequently as the text evolves, and it is expected that your level of comprehension as well as your ability to create proofs will grow as you proceed.

This proves that corresponding entries in $(\mathbf{A}+\mathbf{B})^T$ and $\mathbf{A}^T+\mathbf{B}^T$ are equal, so it must be the case that $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T+\mathbf{B}^T$. Similarly, for each i and j,

$$[(\alpha\mathbf{A})^T]_{ij} = [\alpha\mathbf{A}]_{ji} = \alpha[\mathbf{A}]_{ji} = \alpha[\mathbf{A}^T]_{ij} \implies (\alpha\mathbf{A})^T = \alpha\mathbf{A}^T. \quad \blacksquare$$

Sometimes transposition doesn't change anything. For example, if

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}, \quad \text{then} \quad \mathbf{A}^T = \mathbf{A}.$$

This is because the entries in $\mathbf{A}$ are symmetrically located about the ***main diagonal***—the line from the upper-left-hand corner to the lower-right-hand corner. Matrices of the form $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$ are called ***diagonal matrices***, and they are clearly symmetric in the sense that $\mathbf{D} = \mathbf{D}^T$. This is one of several kinds of symmetries described below.

Symmetries

Let $\mathbf{A} = [a_{ij}]$ be a square matrix.

- $\mathbf{A}$ is said to be a ***symmetric matrix*** whenever $\mathbf{A} = \mathbf{A}^T$, i.e., whenever $a_{ij} = a_{ji}$.
- $\mathbf{A}$ is said to be a ***skew-symmetric matrix*** whenever $\mathbf{A} = -\mathbf{A}^T$, i.e., whenever $a_{ij} = -a_{ji}$.
- $\mathbf{A}$ is said to be a ***hermitian matrix*** whenever $\mathbf{A} = \mathbf{A}^*$, i.e., whenever $a_{ij} = \overline{a}_{ji}$. This is the complex analog of symmetry.
- $\mathbf{A}$ is said to be a ***skew-hermitian matrix*** when $\mathbf{A} = -\mathbf{A}^*$, i.e., whenever $a_{ij} = -\overline{a}_{ji}$. This is the complex analog of skew symmetry.

For example, consider

$$\mathbf{A} = \begin{pmatrix} 1 & 2+4\mathrm{i} & 1-3\mathrm{i} \\ 2-4\mathrm{i} & 3 & 8+6\mathrm{i} \\ 1+3\mathrm{i} & 8-6\mathrm{i} & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2+4\mathrm{i} & 1-3\mathrm{i} \\ 2+4\mathrm{i} & 3 & 8+6\mathrm{i} \\ 1-3\mathrm{i} & 8+6\mathrm{i} & 5 \end{pmatrix}.$$

Can you see that $\mathbf{A}$ is hermitian but not symmetric, while $\mathbf{B}$ is symmetric but not hermitian?

Nature abounds with symmetry, and very often physical symmetry manifests itself as a symmetric matrix in a mathematical model. The following example is an illustration of this principle.

Example 3.2.1

Consider two springs that are connected as shown in Figure 3.2.1.

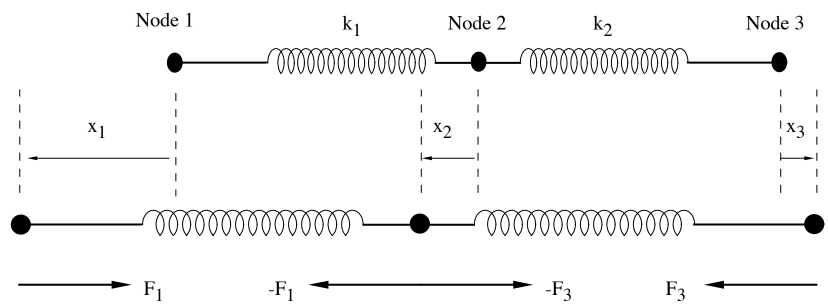

FIGURE 3.2.1

The springs at the top represent the "no tension" position in which no force is being exerted on any of the nodes. Suppose that the springs are stretched or compressed so that the nodes are displaced as indicated in the lower portion of Figure 3.2.1. Stretching or compressing the springs creates a force on each node according to Hooke's law [18] that says that the force exerted by a spring is $F = kx$, where x is the distance the spring is stretched or compressed and where k is a *stiffness constant* inherent to the spring. Suppose our springs have stiffness constants k_1 and k_2, and let F_i be the force on node i when the springs are stretched or compressed. Let's agree that a displacement to the left is positive, while a displacement to the right is negative, and consider a force directed to the right to be positive while one directed to the left is negative. If node 1 is displaced x_1 units, and if node 2 is displaced x_2 units, then the left-hand spring is stretched (or compressed) by a total amount of $x_1 - x_2$ units, so the force on node 1 is

$$F_1 = k_1(x_1 - x_2).$$

Similarly, if node 2 is displaced x_2 units, and if node 3 is displaced x_3 units, then the right-hand spring is stretched by a total amount of $x_2 - x_3$ units, so the force on node 3 is

$$F_3 = -k_2(x_2 - x_3).$$

The minus sign indicates the force is directed to the left. The force on the left-hand side of node 2 is the opposite of the force on node 1, while the force on the right-hand side of node 2 must be the opposite of the force on node 3. That is,

$$F_2 = -F_1 - F_3.$$

---

[18] Hooke's law is named for Robert Hooke (1635–1703), an English physicist, but it was generally known to several people (including Newton) before Hooke's 1678 claim to it was made. Hooke was a creative person who is credited with several inventions, including the wheel barometer, but he was reputed to be a man of "terrible character." This characteristic virtually destroyed his scientific career as well as his personal life. It is said that he lacked mathematical sophistication and that he left much of his work in incomplete form, but he bitterly resented people who built on his ideas by expressing them in terms of elegant mathematical formulations.

Organize the above three equations as a linear system:

$$\begin{aligned} k_1x_1 \qquad\quad - k_1x_2 \qquad\qquad &= F_1, \\ -k_1x_1 + (k_1+k_2)x_2 - k_2x_3 &= F_2, \\ -k_2x_2 + k_2x_3 &= F_3, \end{aligned}$$

and observe that the coefficient matrix, called the ***stiffness matrix,***

$$\mathbf{K} = \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix},$$

is a *symmetric* matrix. The point of this example is that symmetry in the physical problem translates to symmetry in the mathematics by way of the symmetric matrix $\mathbf{K}$. When the two springs are identical (i.e., when $k_1 = k_2 = k$), even more symmetry is present, and in this case

$$\mathbf{K} = k\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Exercises for section 3.2

3.2.1. Determine the unknown quantities in the following expressions.

(a) $3\mathbf{X} = \begin{pmatrix} 0 & 3 \\ 6 & 9 \end{pmatrix}$. (b) $2\begin{pmatrix} x+2 & y+3 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ y & z \end{pmatrix}^T$.

3.2.2. Identify each of the following as symmetric, skew symmetric, or neither.

(a) $\begin{pmatrix} 1 & -3 & 3 \\ -3 & 4 & -3 \\ 3 & 3 & 0 \end{pmatrix}$. (b) $\begin{pmatrix} 0 & -3 & -3 \\ 3 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix}$.

(c) $\begin{pmatrix} 0 & -3 & -3 \\ -3 & 0 & 3 \\ -3 & 3 & 1 \end{pmatrix}$. (d) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$.

3.2.3. Construct an example of a 3×3 matrix $\mathbf{A}$ that satisfies the following conditions.

(a) $\mathbf{A}$ is both symmetric and skew symmetric.
(b) $\mathbf{A}$ is both hermitian and symmetric.
(c) $\mathbf{A}$ is skew hermitian.

3.2.4. Explain why the set of all $n \times n$ symmetric matrices is closed under matrix addition. That is, explain why the sum of two $n \times n$ symmetric matrices is again an $n \times n$ symmetric matrix. Is the set of all $n \times n$ skew-symmetric matrices closed under matrix addition?

3.2.5. Prove that each of the following statements is true.

(a) If $\mathbf{A} = [a_{ij}]$ is skew symmetric, then $a_{jj} = 0$ for each j.

(b) If $\mathbf{A} = [a_{ij}]$ is skew hermitian, then each a_{jj} is a pure imaginary number—i.e., a multiple of the imaginary unit i.

(c) If $\mathbf{A}$ is real and symmetric, then $\mathbf{B} = \mathrm{i}\mathbf{A}$ is skew hermitian.

3.2.6. Let $\mathbf{A}$ be any square matrix.

(a) Show that $\mathbf{A}+\mathbf{A}^T$ is symmetric and $\mathbf{A}-\mathbf{A}^T$ is skew symmetric.

(b) Prove that there is one and only one way to write $\mathbf{A}$ as the sum of a symmetric matrix and a skew-symmetric matrix.

3.2.7. If $\mathbf{A}$ and $\mathbf{B}$ are two matrices of the same shape, prove that each of the following statements is true.

(a) $(\mathbf{A} + \mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*$.

(b) $(\alpha\mathbf{A})^* = \overline{\alpha}\mathbf{A}^*$.

3.2.8. Using the conventions given in Example 3.2.1, determine the stiffness matrix for a system of n identical springs, with stiffness constant k, connected in a line similar to that shown in Figure 3.2.1.

Solutions for Chapter 3

Solutions for exercises in section 3. 2

3.2.1. (a) $\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$ (b) $x = -\frac{1}{2}$, $y = -6$, and $z = 0$

3.2.2. (a) Neither (b) Skew symmetric (c) Symmetric (d) Neither

3.2.3. The 3×3 zero matrix trivially satisfies all conditions, and it is the only possible answer for part (a). The only possible answers for (b) are real symmetric matrices. There are many nontrivial possibilities for (c).

3.2.4. $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{B} = \mathbf{B}^T \implies (\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T = \mathbf{A}+\mathbf{B}$. Yes—the skew-symmetric matrices are also closed under matrix addition.

3.2.5. (a) $\mathbf{A} = -\mathbf{A}^T \implies a_{ij} = -a_{ji}$. If $i = j$, then $a_{jj} = -a_{jj} \implies a_{jj} = 0$.

(b) $\mathbf{A} = -\mathbf{A}^* \implies a_{ij} = -\overline{a_{ji}}$. If $i = j$, then $a_{jj} = -\overline{a_{jj}}$. Write $a_{jj} = x + \mathrm{i}y$ to see that $a_{jj} = -\overline{a_{jj}} \implies x + \mathrm{i}y = -x + \mathrm{i}y \implies x = 0 \implies a_{jj}$ is pure imaginary.

(c) $\mathbf{B}^* = (\mathrm{i}\mathbf{A})^* = -\mathrm{i}\mathbf{A}^* = -\mathrm{i}\overline{\mathbf{A}}^T = -\mathrm{i}\mathbf{A}^T = -\mathrm{i}\mathbf{A} = -\mathbf{B}$.

3.2.6. (a) Let $\mathbf{S} = \mathbf{A}+\mathbf{A}^T$ and $\mathbf{K} = \mathbf{A}-\mathbf{A}^T$. Then $\mathbf{S}^T = \mathbf{A}^T + \mathbf{A}^{T^T} = \mathbf{A}^T + \mathbf{A} = \mathbf{S}$. Likewise, $\mathbf{K}^T = \mathbf{A}^T - \mathbf{A}^{T^T} = \mathbf{A}^T - \mathbf{A} = -\mathbf{K}$.

(b) $\mathbf{A} = \frac{\mathbf{S}}{2} + \frac{\mathbf{K}}{2}$ is one such decomposition. To see it is unique, suppose $\mathbf{A} = \mathbf{X} + \mathbf{Y}$, where $\mathbf{X} = \mathbf{X}^T$ and $\mathbf{Y} = -\mathbf{Y}^T$. Thus, $\mathbf{A}^T = \mathbf{X}^T + \mathbf{Y}^T = \mathbf{X} - \mathbf{Y} \implies \mathbf{A} + \mathbf{A}^T = 2\mathbf{X}$, so that $\mathbf{X} = \frac{\mathbf{A}+\mathbf{A}^T}{2} = \frac{\mathbf{S}}{2}$. A similar argument shows that $\mathbf{Y} = \frac{\mathbf{A}-\mathbf{A}^T}{2} = \frac{\mathbf{K}}{2}$.

3.2.7. (a) $[(\mathbf{A}+\mathbf{B})^*]_{ij} = [\overline{\mathbf{A}+\mathbf{B}}]_{ji} = [\overline{\mathbf{A}} + \overline{\mathbf{B}}]_{ji} = [\overline{\mathbf{A}}]_{ji} + [\overline{\mathbf{B}}]_{ji} = [\mathbf{A}^*]_{ij} + [\mathbf{B}^*]_{ij} = [\mathbf{A}^* + \mathbf{B}^*]_{ij}$

(b) $[(\alpha\mathbf{A})^*]_{ij} = [\overline{\alpha\mathbf{A}}]_{ji} = [\bar{\alpha}\overline{\mathbf{A}}]_{ji} = \bar{\alpha}[\overline{\mathbf{A}}]_{ji} = \bar{\alpha}[\mathbf{A}^*]_{ij}$

3.2.8. $$k\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

Solutions for exercises in section 3. 3

3.3.1. Functions (b) and (f) are linear. For example, to check if (b) is linear, let $\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, and check if $f(\mathbf{A}+\mathbf{B}) = f(\mathbf{A}) + f(\mathbf{B})$ and