3.4 WHY DO IT THIS WAY

If you were given the task of formulating a definition for composing two matrices **A** and **B** in some sort of "natural" multiplicative fashion, your first attempt would probably be to compose **A** and **B** by multiplying corresponding entries—much the same way matrix addition is defined. Asked then to defend the usefulness of such a definition, you might be hard pressed to provide a truly satisfying response. Unless a person is in the right frame of mind, the issue of deciding how to best define matrix multiplication is not at all transparent, especially if it is insisted that the definition be both "natural" and "useful." The world had to wait for Arthur Cayley to come to this proper frame of mind.

As mentioned in §3.1, matrix algebra appeared late in the game. Manipulation on arrays and the theory of determinants existed long before Cayley and his theory of matrices. Perhaps this can be attributed to the fact that the "correct" way to multiply two matrices eluded discovery for such a long time.

Around 1855, Cayley became interested in composing linear functions.[19] In particular, he was investigating linear functions of the type discussed in Example 3.3.2. Typical examples of two such functions are

$$f(\mathbf{x}) = f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} \quad \text{and} \quad g(\mathbf{x}) = g\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{pmatrix}.$$

Consider, as Cayley did, composing f and g to create another linear function

$$h(\mathbf{x}) = f\big(g(\mathbf{x})\big) = f\begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{pmatrix} = \begin{pmatrix} (aA + bC)x_1 + (aB + bD)x_2 \\ (cA + dC)x_1 + (cB + dD)x_2 \end{pmatrix}.$$

It was Cayley's idea to use matrices of coefficients to represent these linear functions. That is, f, g, and h are represented by

$$\mathbf{F} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{and} \quad \mathbf{H} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \end{pmatrix}.$$

After making this association, it was only natural for Cayley to call **H** the *composition* (or *product*) of **F** and **G**, and to write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \end{pmatrix}. \tag{3.4.1}$$

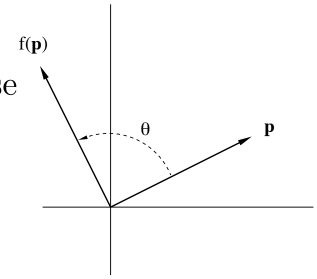
In other words, the product of two matrices represents the composition of the two associated linear functions. By means of this observation, Cayley brought to life the subjects of matrix analysis and linear algebra.

[19] Cayley was not the first to compose linear functions. In fact, Gauss used these compositions as early as 1801, but not in the form of an array of coefficients. Cayley was the first to make the connection between composition of linear functions and the composition of the associated matrices. Cayley's work from 1855 to 1857 is regarded as being the birth of our subject.

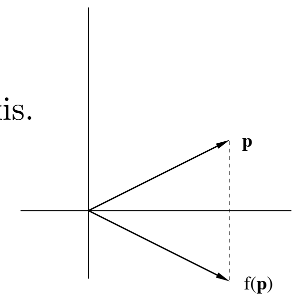
Exercises for section 3.4

Each problem in this section concerns the following three linear transformations in $\Re^2$.

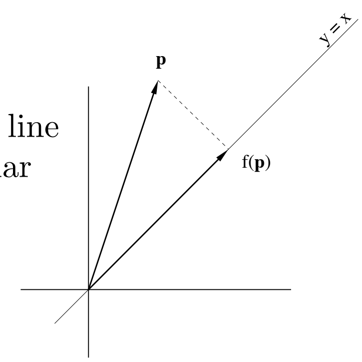
Rotation: Rotate points counterclockwise through an angle θ.



Reflection: Reflect points about the x-axis.



Projection: Project points onto the line $y = x$ in a perpendicular manner.



3.4.1. Determine the matrix associated with each of these linear functions. That is, determine the a_{ij}'s such that

$$f(\mathbf{p}) = f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}.$$

3.4.2. By using matrix multiplication, determine the linear function obtained by performing a rotation followed by a reflection.

3.4.3. By using matrix multiplication, determine the linear function obtained by first performing a reflection, then a rotation, and finally a projection.

Solutions for exercises in section 3. 4

3.4.1. Refer to the solution for Exercise 3.3.4. If $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{P}$ denote the matrices associated with the rotation, reflection, and projection, respectively, then

$$\mathbf{Q} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

3.4.2. Refer to the solution for Exercise 3.4.1 and write

$$\mathbf{RQ} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}.$$

If $Q(\mathbf{x})$ is the rotation function and $R(\mathbf{x})$ is the reflection function, then the composition is

$$R\big(Q(\mathbf{x})\big) = \begin{pmatrix} (\cos\theta)x_1 - (\sin\theta)x_2 \\ -(\sin\theta)x_1 - (\cos\theta)x_2 \end{pmatrix}.$$

3.4.3. Refer to the solution for Exercise 3.4.1 and write

$$\begin{aligned} \mathbf{PQR} &= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \cos\theta + \sin\theta & \sin\theta - \cos\theta \\ \cos\theta + \sin\theta & \sin\theta - \cos\theta \end{pmatrix}. \end{aligned}$$

Therefore, the composition of the three functions in the order asked for is

$$P\Big(Q\big(R(\mathbf{x})\big)\Big) = \frac{1}{2} \begin{pmatrix} (\cos\theta + \sin\theta)x_1 + (\sin\theta - \cos\theta)x_2 \\ (\cos\theta + \sin\theta)x_1 + (\sin\theta - \cos\theta)x_2 \end{pmatrix}.$$

Solutions for exercises in section 3. 5

3.5.1. (a) $\mathbf{AB} = \begin{pmatrix} 10 & 15 \\ 12 & 8 \\ 28 & 52 \end{pmatrix}$ (b) $\mathbf{BA}$ does not exist (c) $\mathbf{CB}$ does not exist

(d) $\mathbf{C}^T\mathbf{B} = \begin{pmatrix} 10 & 31 \end{pmatrix}$ (e) $\mathbf{A}^2 = \begin{pmatrix} 13 & -1 & 19 \\ 16 & 13 & 12 \\ 36 & -17 & 64 \end{pmatrix}$ (f) $\mathbf{B}^2$ does not exist

(g) $\mathbf{C}^T\mathbf{C} = 14$ (h) $\mathbf{CC}^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ (i) $\mathbf{BB}^T = \begin{pmatrix} 5 & 8 & 17 \\ 8 & 16 & 28 \\ 17 & 28 & 58 \end{pmatrix}$

(j) $\mathbf{B}^T\mathbf{B} = \begin{pmatrix} 10 & 23 \\ 23 & 69 \end{pmatrix}$ (k) $\mathbf{C}^T\mathbf{AC} = 76$