3.5 MATRIX MULTIPLICATION

The purpose of this section is to further develop the concept of matrix multiplication as introduced in the previous section. In order to do this, it is helpful to begin by composing a single row with a single column. If

$$\mathbf{R} = \begin{pmatrix} r_1 & r_2 & \cdots & r_n \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

the ***standard inner product*** of $\mathbf{R}$ with $\mathbf{C}$ is defined to be the scalar

$$\mathbf{RC} = r_1c_1 + r_2c_2 + \cdots + r_nc_n = \sum_{i=1}^{n} r_ic_i.$$

For example,

$$\begin{pmatrix} 2 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (2)(1) + (4)(2) + (-2)(3) = 4.$$

Recall from (3.4.1) that the product of two 2×2 matrices

$$\mathbf{F} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

was defined naturally by writing

$$\mathbf{FG} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \end{pmatrix} = \mathbf{H}.$$

Notice that the (i, j)-entry in the product $\mathbf{H}$ can be described as the inner product of the i^{th} row of $\mathbf{F}$ with the j^{th} column in $\mathbf{G}$. That is,

$$h_{11} = \mathbf{F}_{1*}\mathbf{G}_{*1} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}, \qquad h_{12} = \mathbf{F}_{1*}\mathbf{G}_{*2} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix},$$

$$h_{21} = \mathbf{F}_{2*}\mathbf{G}_{*1} = \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}, \qquad h_{22} = \mathbf{F}_{2*}\mathbf{G}_{*2} = \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix}.$$

This is exactly the way that the general definition of matrix multiplication is formulated.

Matrix Multiplication

- Matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be ***conformable*** for multiplication in the order $\mathbf{AB}$ whenever $\mathbf{A}$ has exactly as many columns as $\mathbf{B}$ has rows—i.e., $\mathbf{A}$ is $m \times p$ and $\mathbf{B}$ is $p \times n$.

- For conformable matrices $\mathbf{A}_{m\times p} = [a_{ij}]$ and $\mathbf{B}_{p\times n} = [b_{ij}]$, the ***matrix product*** $\mathbf{AB}$ is defined to be the $m \times n$ matrix whose (i,j)-entry is the inner product of the i^{th} row of $\mathbf{A}$ with the j^{th} column in $\mathbf{B}$. That is,

$$[\mathbf{AB}]_{ij} = \mathbf{A}_{i*}\mathbf{B}_{*j} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$

- In case $\mathbf{A}$ and $\mathbf{B}$ fail to be conformable—i.e., $\mathbf{A}$ is $m \times p$ and $\mathbf{B}$ is $q \times n$ with $p \neq q$—then no product $\mathbf{AB}$ is defined.

For example, if

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{2\times 3} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{pmatrix}_{3\times 4}$$

inside ones match

shape of the product

then the product $\mathbf{AB}$ exists and has shape 2×4. Consider a typical entry of this product, say, the (2,3)-entry. The definition says $[\mathbf{AB}]_{23}$ is obtained by forming the inner product of the second row of $\mathbf{A}$ with the third column of $\mathbf{B}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21} \quad a_{22} \quad a_{23}} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{pmatrix},$$

so

$$[\mathbf{AB}]_{23} = \mathbf{A}_{2*}\mathbf{B}_{*3} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} = \sum_{k=1}^{3} a_{2k}b_{k3}.$$

For example,

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -4 \\ -3 & 0 & 5 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 & 3 & -3 & 2 \\ 2 & 5 & -1 & 8 \\ -1 & 2 & 0 & 2 \end{pmatrix} \Longrightarrow \mathbf{AB} = \begin{pmatrix} 8 & 3 & -7 & 4 \\ -8 & 1 & 9 & 4 \end{pmatrix}.$$

Notice that in spite of the fact that the product $\mathbf{AB}$ exists, the product $\mathbf{BA}$ is not defined—matrix $\mathbf{B}$ is 3×4 and $\mathbf{A}$ is 2×3, and the inside dimensions don't match in this order. Even when the products $\mathbf{AB}$ and $\mathbf{BA}$ each exist and have the same shape, they need not be equal. For example,

$$\mathbf{A}=\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{B}=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Longrightarrow \mathbf{AB}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{BA}=\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}. \quad (3.5.1)$$

This disturbing feature is a primary difference between scalar and matrix algebra.

Matrix Multiplication Is Not Commutative

Matrix multiplication is a noncommutative operation—i.e., it is possible for $\mathbf{AB} \neq \mathbf{BA}$, even when both products exist and have the same shape.

There are other major differences between multiplication of matrices and multiplication of scalars. For scalars,

$$\alpha\beta = 0 \quad \text{implies} \quad \alpha = 0 \quad \text{or} \quad \beta = 0. \qquad (3.5.2)$$

However, the analogous statement for matrices does not hold—the matrices given in (3.5.1) show that it is possible for $\mathbf{AB} = \mathbf{0}$ with $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$. Related to this issue is a rule sometimes known as the ***cancellation law.*** For scalars, this law says that

$$\alpha\beta = \alpha\gamma \quad \text{and} \quad \alpha \neq 0 \quad \text{implies} \quad \beta = \gamma. \qquad (3.5.3)$$

This is true because we invoke (3.5.2) to deduce that $\alpha(\beta - \gamma) = 0$ implies $\beta - \gamma = 0$. Since (3.5.2) does not hold for matrices, we cannot expect (3.5.3) to hold for matrices.

Example 3.5.1

The cancellation law (3.5.3) fails for matrix multiplication. If

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix},$$

then

$$\mathbf{AB} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \mathbf{AC} \quad \text{but} \quad \mathbf{B} \neq \mathbf{C}$$

in spite of the fact that $\mathbf{A} \neq \mathbf{0}$.

There are various ways to express the individual rows and columns of a matrix product. For example, the i^{th} row of $\mathbf{AB}$ is

$$[\mathbf{AB}]_{i*} = \big[\mathbf{A}_{i*}\mathbf{B}_{*1} \,|\, \mathbf{A}_{i*}\mathbf{B}_{*2} \,|\, \cdots \,|\, \mathbf{A}_{i*}\mathbf{B}_{*n}\big] = \mathbf{A}_{i*}\mathbf{B}$$
$$= \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{ip} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{1*} \\ \mathbf{B}_{2*} \\ \vdots \\ \mathbf{B}_{p*} \end{pmatrix} = a_{i1}\mathbf{B}_{1*} + a_{i2}\mathbf{B}_{2*} + \cdots + a_{ip}\mathbf{B}_{p*}.$$

As shown below, there are similar representations for the individual columns.

Rows and Columns of a Product

Suppose that $\mathbf{A} = [a_{ij}]$ is $m \times p$ and $\mathbf{B} = [b_{ij}]$ is $p \times n$.

- $[\mathbf{AB}]_{i*} = \mathbf{A}_{i*}\mathbf{B}$ $\big[$(i^{th} row of $\mathbf{AB}$)=(i^{th} row of $\mathbf{A}$) $\times\mathbf{B}\big]$. (3.5.4)
- $[\mathbf{AB}]_{*j} = \mathbf{AB}_{*j}$ $\big[$(j^{th} col of $\mathbf{AB}$)= $\mathbf{A}\times$ (j^{th} col of $\mathbf{B}$)$\big]$. (3.5.5)
- $[\mathbf{AB}]_{i*} = a_{i1}\mathbf{B}_{1*} + a_{i2}\mathbf{B}_{2*} + \cdots + a_{ip}\mathbf{B}_{p*} = \sum_{k=1}^{p} a_{ik}\mathbf{B}_{k*}$. (3.5.6)
- $[\mathbf{AB}]_{*j} = \mathbf{A}_{*1}b_{1j} + \mathbf{A}_{*2}b_{2j} + \cdots + \mathbf{A}_{*p}b_{pj} = \sum_{k=1}^{p} \mathbf{A}_{*k}b_{kj}$. (3.5.7)

These last two equations show that rows of $\mathbf{AB}$ are combinations of rows of $\mathbf{B}$, while columns of $\mathbf{AB}$ are combinations of columns of $\mathbf{A}$.

For example, if $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -4 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -5 & 1 \\ 2 & -7 & 2 \\ 1 & -2 & 0 \end{pmatrix}$, then the second row of $\mathbf{AB}$ is

$$[\mathbf{AB}]_{2*} = \mathbf{A}_{2*}\mathbf{B} = \begin{pmatrix} 3 & -4 & 5 \end{pmatrix} \begin{pmatrix} 3 & -5 & 1 \\ 2 & -7 & 2 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 3 & -5 \end{pmatrix},$$

and the second column of $\mathbf{AB}$ is

$$[\mathbf{AB}]_{*2} = \mathbf{AB}_{*2} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -4 & 5 \end{pmatrix} \begin{pmatrix} -5 \\ -7 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}.$$

This example makes the point that it is wasted effort to compute the entire product if only one row or column is called for. Although it's not necessary to compute the complete product, you may wish to verify that

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -4 & 5 \end{pmatrix} \begin{pmatrix} 3 & -5 & 1 \\ 2 & -7 & 2 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 9 & -3 \\ 6 & 3 & -5 \end{pmatrix}.$$

Matrix multiplication provides a convenient representation for a linear system of equations. For example, the 3×4 system

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + 8x_4 &= 7, \\ 3x_1 + 5x_2 + 6x_3 + 2x_4 &= 6, \\ 4x_1 + 2x_2 + 4x_3 + 9x_4 &= 4, \end{aligned}$$

can be written as $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A}_{3\times 4} = \begin{pmatrix} 2 & 3 & 4 & 8 \\ 3 & 5 & 6 & 2 \\ 4 & 2 & 4 & 9 \end{pmatrix}, \quad \mathbf{x}_{4\times 1} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_{3\times 1} = \begin{pmatrix} 7 \\ 6 \\ 4 \end{pmatrix}.$$

And this example generalizes to become the following statement.

Linear Systems

Every linear system of m equations in n unknowns

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m, \end{array}$$

can be written as a single matrix equation $\mathbf{Ax} = \mathbf{b}$ in which

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Conversely, every matrix equation of the form $\mathbf{A}_{m\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{m\times 1}$ represents a system of m linear equations in n unknowns.

The numerical solution of a linear system was presented earlier in the text without the aid of matrix multiplication because the operation of matrix multiplication is not an integral part of the arithmetical process used to extract a solution by means of Gaussian elimination. Viewing a linear system as a single matrix equation $\mathbf{Ax} = \mathbf{b}$ is more of a notational convenience that can be used to uncover theoretical properties and to prove general theorems concerning linear systems.

For example, a very concise proof of the fact (2.3.5) stating that a system of equations $\mathbf{A}_{m\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{m\times 1}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns in $\mathbf{A}$ is obtained by noting that the system is consistent if and only if there exists a column $\mathbf{s}$ that satisfies

$$\mathbf{b} = \mathbf{A}\mathbf{s} = \begin{pmatrix} \mathbf{A}_{*1} & \mathbf{A}_{*2} & \cdots & \mathbf{A}_{*n} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} = \mathbf{A}_{*1}s_1 + \mathbf{A}_{*2}s_2 + \cdots + \mathbf{A}_{*n}s_n.$$

The following example illustrates a common situation in which matrix multiplication arises naturally.

Example 3.5.2

An airline serves five cities, say, A, B, C, D, and H, in which H is the "hub city." The various routes between the cities are indicated in Figure 3.5.1.

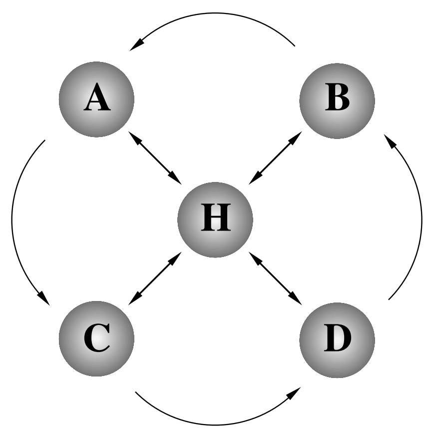

FIGURE 3.5.1

Suppose you wish to travel from city A to city B so that at least two connecting flights are required to make the trip. Flights $(A \to H)$ and $(H \to B)$ provide the minimal number of connections. However, if space on either of these two flights is not available, you will have to make at least three flights. Several questions arise. How many routes from city A to city B require *exactly* three connecting flights? How many routes require *no more than* four flights—and so forth? Since this particular network is small, these questions can be answered by "eyeballing" the diagram, but the "eyeball method" won't get you very far with the large networks that occur in more practical situations. Let's see how matrix algebra can be applied. Begin by creating a ***connectivity matrix*** $\mathbf{C} = [c_{ij}]$ (also known as an ***adjacency matrix***) in which

$$c_{ij} = \begin{cases} 1 & \text{if there is a flight from city } i \text{ to city } j, \\ 0 & \text{otherwise.} \end{cases}$$

For the network depicted in Figure 3.5.1,

$$\mathbf{C} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ H \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & H \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

The matrix $\mathbf{C}$ together with its powers $\mathbf{C}^2, \mathbf{C}^3, \mathbf{C}^4, \ldots$ will provide all of the information needed to analyze the network. To see how, notice that since c_{ik} is the number of direct routes from city i to city k, and since c_{kj} is the number of direct routes from city k to city j, it follows that $c_{ik}c_{kj}$ must be the number of 2-flight routes from city i to city j that have a connection at city k. Consequently, the (i,j)-entry in the product $\mathbf{C}^2 = \mathbf{CC}$ is

$$[\mathbf{C}^2]_{ij} = \sum_{k=1}^{5} c_{ik}c_{kj} = \text{ the total number of 2-flight routes from city } i \text{ to city } j.$$

Similarly, the (i,j)-entry in the product $\mathbf{C}^3 = \mathbf{CCC}$ is

$$[\mathbf{C}^3]_{ij} = \sum_{k_1,k_2=1}^{5} c_{ik_1}c_{k_1k_2}c_{k_2j} = \text{ number of 3-flight routes from city } i \text{ to city } j,$$

and, in general,

$$[\mathbf{C}^n]_{ij} = \sum_{k_1,k_2,\cdots,k_{n-1}=1}^{5} c_{ik_1}c_{k_1k_2}\cdots c_{k_{n-2}k_{n-1}}c_{k_{n-1}j}$$

is the total number of n-flight routes from city i to city j. Therefore, the total number of routes from city i to city j that require *no more* than n flights must be given by

$$[\mathbf{C}]_{ij} + [\mathbf{C}^2]_{ij} + [\mathbf{C}^3]_{ij} + \cdots + [\mathbf{C}^n]_{ij} = [\mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \cdots + \mathbf{C}^n]_{ij}.$$

For our particular network,

$$\mathbf{C}^2 = \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}, \quad \mathbf{C}^3 = \begin{pmatrix} 2 & 3 & 2 & 2 & 5 \\ 2 & 2 & 2 & 3 & 5 \\ 3 & 2 & 2 & 2 & 5 \\ 2 & 2 & 3 & 2 & 5 \\ 5 & 5 & 5 & 5 & 4 \end{pmatrix}, \quad \mathbf{C}^4 = \begin{pmatrix} 8 & 7 & 7 & 7 & 9 \\ 7 & 8 & 7 & 7 & 9 \\ 7 & 7 & 8 & 7 & 9 \\ 7 & 7 & 7 & 8 & 9 \\ 9 & 9 & 9 & 9 & 20 \end{pmatrix},$$

and

$$\mathbf{C}+\mathbf{C}^2+\mathbf{C}^3+\mathbf{C}^4=\begin{pmatrix}11&11&11&11&16\\11&11&11&11&16\\11&11&11&11&16\\11&11&11&11&16\\16&16&16&16&28\end{pmatrix}.$$

The fact that $[\mathbf{C}^3]_{12}=3$ means there are exactly 3 three-flight routes from city A to city B, and $[\mathbf{C}^4]_{12}=7$ means there are exactly 7 four-flight routes—try to identify them. Furthermore, $[\mathbf{C}+\mathbf{C}^2+\mathbf{C}^3+\mathbf{C}^4]_{12}=11$ means there are 11 routes from city A to city B that require no more than 4 flights.

Exercises for section 3.5

3.5.1. For $\mathbf{A}=\begin{pmatrix}1&-2&3\\0&-5&4\\4&-3&8\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1&2\\0&4\\3&7\end{pmatrix}$, and $\mathbf{C}=\begin{pmatrix}1\\2\\3\end{pmatrix}$, compute the following products when possible.
(a) $\mathbf{AB}$, (b) $\mathbf{BA}$, (c) $\mathbf{CB}$, (d) $\mathbf{C}^T\mathbf{B}$, (e) $\mathbf{A}^2$, (f) $\mathbf{B}^2$,
(g) $\mathbf{C}^T\mathbf{C}$, (h) $\mathbf{CC}^T$, (i) $\mathbf{BB}^T$, (j) $\mathbf{B}^T\mathbf{B}$, (k) $\mathbf{C}^T\mathbf{AC}$.

3.5.2. Consider the following system of equations:

$$\begin{aligned}2x_1 + x_2 + x_3 &= 3,\\ 4x_1 + 2x_3 &= 10,\\ 2x_1 + 2x_2 &= -2.\end{aligned}$$

(a) Write the system as a matrix equation of the form $\mathbf{Ax}=\mathbf{b}$.
(b) Write the solution of the system as a column $\mathbf{s}$ and verify by matrix multiplication that $\mathbf{s}$ satisfies the equation $\mathbf{Ax}=\mathbf{b}$.
(c) Write $\mathbf{b}$ as a linear combination of the columns in $\mathbf{A}$.

3.5.3. Let $\mathbf{E}=\begin{pmatrix}1&0&0\\0&1&0\\3&0&1\end{pmatrix}$ and let $\mathbf{A}$ be an arbitrary 3×3 matrix.
(a) Describe the rows of $\mathbf{EA}$ in terms of the rows of $\mathbf{A}$.
(b) Describe the columns of $\mathbf{AE}$ in terms of the columns of $\mathbf{A}$.

3.5.4. Let $\mathbf{e}_j$ denote the j^{th} ***unit column*** that contains a 1 in the j^{th} position and zeros everywhere else. For a general matrix $\mathbf{A}_{n\times n}$, describe the following products. (a) $\mathbf{Ae}_j$ (b) $\mathbf{e}_i^T\mathbf{A}$ (c) $\mathbf{e}_i^T\mathbf{Ae}_j$

3.5.5. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices. If $\mathbf{Ax} = \mathbf{Bx}$ holds for all $n \times 1$ columns $\mathbf{x}$, prove that $\mathbf{A} = \mathbf{B}$. **Hint:** What happens when $\mathbf{x}$ is a unit column?

3.5.6. For $\mathbf{A} = \begin{pmatrix} 1/2 & \alpha \\ 0 & 1/2 \end{pmatrix}$, determine $\lim_{n\to\infty} \mathbf{A}^n$. **Hint:** Compute a few powers of $\mathbf{A}$ and try to deduce the general form of $\mathbf{A}^n$.

3.5.7. If $\mathbf{C}_{m\times 1}$ and $\mathbf{R}_{1\times n}$ are matrices consisting of a single column and a single row, respectively, then the matrix product $\mathbf{P}_{m\times n} = \mathbf{CR}$ is sometimes called the ***outer product*** of $\mathbf{C}$ with $\mathbf{R}$. For conformable matrices $\mathbf{A}$ and $\mathbf{B}$, explain how to write the product $\mathbf{AB}$ as a sum of outer products involving the columns of $\mathbf{A}$ and the rows of $\mathbf{B}$.

3.5.8. A square matrix $\mathbf{U} = [u_{ij}]$ is said to be ***upper triangular*** whenever $u_{ij} = 0$ for $i > j$—i.e., all entries below the main diagonal are 0.

(a) If $\mathbf{A}$ and $\mathbf{B}$ are two $n \times n$ upper-triangular matrices, explain why the product $\mathbf{AB}$ must also be upper triangular.

(b) If $\mathbf{A}_{n\times n}$ and $\mathbf{B}_{n\times n}$ are upper triangular, what are the diagonal entries of $\mathbf{AB}$?

(c) $\mathbf{L}$ is ***lower triangular*** when $\ell_{ij} = 0$ for $i < j$. Is it true that the product of two $n \times n$ lower-triangular matrices is again lower triangular?

3.5.9. If $\mathbf{A} = [a_{ij}(t)]$ is a matrix whose entries are functions of a variable t, the ***derivative*** of $\mathbf{A}$ with respect to t is defined to be the matrix of derivatives. That is,

$$\frac{d\mathbf{A}}{dt} = \left[\frac{da_{ij}}{dt}\right].$$

Derive the *product rule for differentiation*

$$\frac{d(\mathbf{AB})}{dt} = \frac{d\mathbf{A}}{dt}\mathbf{B} + \mathbf{A}\frac{d\mathbf{B}}{dt}.$$

3.5.10. Let $\mathbf{C}_{n\times n}$ be the connectivity matrix associated with a network of n nodes such as that described in Example 3.5.2, and let $\mathbf{e}$ be the $n \times 1$ column of all 1's. In terms of the network, describe the entries in each of the following products.

(a) Interpret the product $\mathbf{Ce}$.

(b) Interpret the product $\mathbf{e}^T\mathbf{C}$.

3.5.11. Consider three tanks each containing V gallons of brine. The tanks are connected as shown in Figure 3.5.2, and all spigots are opened at once. As fresh water at the rate of r gal/sec is pumped into the top of the first tank, r gal/sec leaves from the bottom and flows into the next tank, and so on down the line—there are r gal/sec entering at the top and leaving through the bottom of each tank.

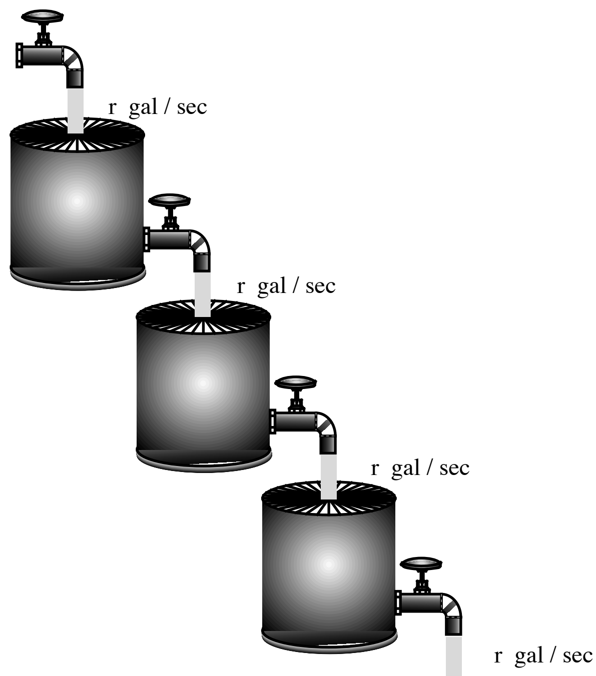

FIGURE 3.5.2

Let $x_i(t)$ denote the number of pounds of salt in tank i at time t, and let

$$\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \quad \text{and} \quad \frac{d\mathbf{x}}{dt} = \begin{pmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{pmatrix}.$$

Assuming that complete mixing occurs in each tank on a continuous basis, show that

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}, \quad \text{where} \quad \mathbf{A} = \frac{r}{V}\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Hint: Use the fact that

$$\frac{dx_i}{dt} = \text{rate of change} = \frac{\text{lbs}}{\text{sec}}\text{ coming in} - \frac{\text{lbs}}{\text{sec}}\text{ going out.}$$

Solutions for exercises in section 3. 4

3.4.1. Refer to the solution for Exercise 3.3.4. If $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{P}$ denote the matrices associated with the rotation, reflection, and projection, respectively, then

$$\mathbf{Q} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

3.4.2. Refer to the solution for Exercise 3.4.1 and write

$$\mathbf{RQ} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}.$$

If $Q(\mathbf{x})$ is the rotation function and $R(\mathbf{x})$ is the reflection function, then the composition is

$$R\big(Q(\mathbf{x})\big) = \begin{pmatrix} (\cos\theta)x_1 - (\sin\theta)x_2 \\ -(\sin\theta)x_1 - (\cos\theta)x_2 \end{pmatrix}.$$

3.4.3. Refer to the solution for Exercise 3.4.1 and write

$$\begin{aligned} \mathbf{PQR} &= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \cos\theta + \sin\theta & \sin\theta - \cos\theta \\ \cos\theta + \sin\theta & \sin\theta - \cos\theta \end{pmatrix}. \end{aligned}$$

Therefore, the composition of the three functions in the order asked for is

$$P\Big(Q\big(R(\mathbf{x})\big)\Big) = \frac{1}{2}\begin{pmatrix} (\cos\theta + \sin\theta)x_1 + (\sin\theta - \cos\theta)x_2 \\ (\cos\theta + \sin\theta)x_1 + (\sin\theta - \cos\theta)x_2 \end{pmatrix}.$$

Solutions for exercises in section 3. 5

3.5.1. (a) $\mathbf{AB} = \begin{pmatrix} 10 & 15 \\ 12 & 8 \\ 28 & 52 \end{pmatrix}$ (b) $\mathbf{BA}$ does not exist (c) $\mathbf{CB}$ does not exist

(d) $\mathbf{C}^T\mathbf{B} = \begin{pmatrix} 10 & 31 \end{pmatrix}$ (e) $\mathbf{A}^2 = \begin{pmatrix} 13 & -1 & 19 \\ 16 & 13 & 12 \\ 36 & -17 & 64 \end{pmatrix}$ (f) $\mathbf{B}^2$ does not exist

(g) $\mathbf{C}^T\mathbf{C} = 14$ (h) $\mathbf{CC}^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ (i) $\mathbf{BB}^T = \begin{pmatrix} 5 & 8 & 17 \\ 8 & 16 & 28 \\ 17 & 28 & 58 \end{pmatrix}$

(j) $\mathbf{B}^T\mathbf{B} = \begin{pmatrix} 10 & 23 \\ 23 & 69 \end{pmatrix}$ (k) $\mathbf{C}^T\mathbf{AC} = 76$

3.5.2. (a) $\mathbf{A}=\begin{pmatrix}2&1&1\\4&0&2\\2&2&0\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},\ \mathbf{b}=\begin{pmatrix}3\\10\\-2\end{pmatrix}$ (b) $\mathbf{s}=\begin{pmatrix}1\\-2\\3\end{pmatrix}$

(c) $\mathbf{b}=\mathbf{A}_{*1}-2\mathbf{A}_{*2}+3\mathbf{A}_{*3}$

3.5.3. (a) $\mathbf{EA}=\begin{pmatrix}\mathbf{A}_{1*}\\\mathbf{A}_{2*}\\3\mathbf{A}_{1*}+\mathbf{A}_{3*}\end{pmatrix}$ (b) $\mathbf{AE}=\begin{pmatrix}\mathbf{A}_{*1}+3\mathbf{A}_{*3} & \mathbf{A}_{*2} & \mathbf{A}_{*3}\end{pmatrix}$

3.5.4. (a) $\mathbf{A}_{*j}$ (b) $\mathbf{A}_{i*}$ (c) a_{ij}

3.5.5. $\mathbf{Ax}=\mathbf{Bx}\ \forall\ \mathbf{x}\implies\mathbf{Ae}_j=\mathbf{Be}_j\ \forall\ \mathbf{e}_j\implies\mathbf{A}_{*j}=\mathbf{B}_{*j}\ \forall\ j\implies\mathbf{A}=\mathbf{B}$. (The symbol $\forall$ is mathematical shorthand for the phrase "for all.")

3.5.6. The limit is the zero matrix.

3.5.7. If $\mathbf{A}$ is $m\times p$ and $\mathbf{B}$ is $p\times n$, write the product as

$$\mathbf{AB}=\begin{pmatrix}\mathbf{A}_{*1} & \mathbf{A}_{*2} & \cdots & \mathbf{A}_{*p}\end{pmatrix}\begin{pmatrix}\mathbf{B}_{1*}\\\mathbf{B}_{2*}\\\vdots\\\mathbf{B}_{p*}\end{pmatrix}=\mathbf{A}_{*1}\mathbf{B}_{1*}+\mathbf{A}_{*2}\mathbf{B}_{2*}+\cdots+\mathbf{A}_{*p}\mathbf{B}_{p*}$$
$$=\sum_{k=1}^{p}\mathbf{A}_{*k}\mathbf{B}_{k*}.$$

3.5.8. (a) $[\mathbf{AB}]_{ij}=\mathbf{A}_{i*}\mathbf{B}_{*j}=\begin{pmatrix}0&\cdots&0&a_{ii}&\cdots&a_{in}\end{pmatrix}\begin{pmatrix}b_{1j}\\\vdots\\b_{jj}\\0\\\vdots\\0\end{pmatrix}$ is 0 when $i>j$.

(b) When $i=j$, the only nonzero term in the product $\mathbf{A}_{i*}\mathbf{B}_{*i}$ is $a_{ii}b_{ii}$.

(c) Yes.

3.5.9. Use $[\mathbf{AB}]_{ij}=\sum_k a_{ik}b_{kj}$ along with the rules of differentiation to write

$$\frac{d[\mathbf{AB}]_{ij}}{dt}=\frac{d\left(\sum_k a_{ik}b_{kj}\right)}{dt}=\sum_k\frac{d(a_{ik}b_{kj})}{dt}$$
$$=\sum_k\left(\frac{da_{ik}}{dt}b_{kj}+a_{ik}\frac{db_{kj}}{dt}\right)=\sum_k\frac{da_{ik}}{dt}b_{kj}+\sum_k a_{ik}\frac{db_{kj}}{dt}$$
$$=\left[\frac{d\mathbf{A}}{dt}\mathbf{B}\right]_{ij}+\left[\mathbf{A}\frac{d\mathbf{B}}{dt}\right]_{ij}=\left[\frac{d\mathbf{A}}{dt}\mathbf{B}+\mathbf{A}\frac{d\mathbf{B}}{dt}\right]_{ij}.$$

3.5.10. (a) $[\mathbf{Ce}]_i=$ the total number of paths *leaving* node i.

(b) $[\mathbf{e}^T\mathbf{C}]_i=$ the total number of paths *entering* node i.

3.5.11. At time t, the concentration of salt in tank i is $\frac{x_i(t)}{V}$ lbs/gal. For tank 1,

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{\text{lbs}}{\text{sec}} \text{ coming in} - \frac{\text{lbs}}{\text{sec}} \text{ going out} = 0\frac{\text{lbs}}{\text{sec}} - \left(r\frac{\text{gal}}{\text{sec}} \times \frac{x_1(t)}{V}\frac{\text{lbs}}{\text{gal}}\right) \\ &= -\frac{r}{V}x_1(t)\frac{\text{lbs}}{\text{sec}}.\end{aligned}$$

For tank 2,

$$\begin{aligned}\frac{dx_2}{dt} &= \frac{\text{lbs}}{\text{sec}} \text{ coming in} - \frac{\text{lbs}}{\text{sec}} \text{ going out} = \frac{r}{V}x_1(t)\frac{\text{lbs}}{\text{sec}} - \left(r\frac{\text{gal}}{\text{sec}} \times \frac{x_2(t)}{V}\frac{\text{lbs}}{\text{gal}}\right) \\ &= \frac{r}{V}x_1(t)\frac{\text{lbs}}{\text{sec}} - \frac{r}{V}x_2(t)\frac{\text{lbs}}{\text{sec}} = \frac{r}{V}\Big(x_1(t) - x_2(t)\Big),\end{aligned}$$

and for tank 3,

$$\begin{aligned}\frac{dx_3}{dt} &= \frac{\text{lbs}}{\text{sec}} \text{ coming in} - \frac{\text{lbs}}{\text{sec}} \text{ going out} = \frac{r}{V}x_2(t)\frac{\text{lbs}}{\text{sec}} - \left(r\frac{\text{gal}}{\text{sec}} \times \frac{x_3(t)}{V}\frac{\text{lbs}}{\text{gal}}\right) \\ &= \frac{r}{V}x_2(t)\frac{\text{lbs}}{\text{sec}} - \frac{r}{V}x_3(t)\frac{\text{lbs}}{\text{sec}} = \frac{r}{V}\Big(x_2(t) - x_3(t)\Big).\end{aligned}$$

This is a system of three linear first-order differential equations

$$\begin{aligned}\frac{dx_1}{dt} &= \tfrac{r}{V}\Big(-x_1(t)\Big) \\ \frac{dx_2}{dt} &= \tfrac{r}{V}\Big(x_1(t) - x_2(t)\Big) \\ \frac{dx_3}{dt} &= \tfrac{r}{V}\Big(x_2(t) - x_3(t)\Big)\end{aligned}$$

that can be written as a single matrix differential equation

$$\begin{pmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \end{pmatrix} = \frac{r}{V}\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$