3.9 ELEMENTARY MATRICES AND EQUIVALENCE

A common theme in mathematics is to break complicated objects into more elementary components, such as factoring large polynomials into products of smaller polynomials. The purpose of this section is to lay the groundwork for similar ideas in matrix algebra by considering how a general matrix might be factored into a product of more "elementary" matrices.

Elementary Matrices

Matrices of the form $\mathbf{I}-\mathbf{u}\mathbf{v}^T$, where $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ columns such that $\mathbf{v}^T\mathbf{u} \neq 1$ are called ***elementary matrices***, and we know from (3.8.1) that all such matrices are nonsingular and

$$\left(\mathbf{I}-\mathbf{u}\mathbf{v}^T\right)^{-1} = \mathbf{I} - \frac{\mathbf{u}\mathbf{v}^T}{\mathbf{v}^T\mathbf{u}-1}. \tag{3.9.1}$$

Notice that inverses of elementary matrices are elementary matrices.

We are primarily interested in the elementary matrices associated with the three elementary row (or column) operations hereafter referred to as follows.

- Type I is interchanging rows (columns) i and j.
- Type II is multiplying row (column) i by $\alpha \neq 0$.
- Type III is adding a multiple of row (column) i to row (column) j.

An elementary matrix of Type I, II, or III is created by performing an elementary operation of Type I, II, or III to an identity matrix. For example, the matrices

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix} \tag{3.9.2}$$

are elementary matrices of Types I, II, and III, respectively, because $\mathbf{E}_1$ arises by interchanging rows 1 and 2 in $\mathbf{I}_3$, whereas $\mathbf{E}_2$ is generated by multiplying row 2 in $\mathbf{I}_3$ by α, and $\mathbf{E}_3$ is constructed by multiplying row 1 in $\mathbf{I}_3$ by α and adding the result to row 3. The matrices in (3.9.2) also can be generated by column operations. For example, $\mathbf{E}_3$ can be obtained by adding α times the third column of $\mathbf{I}_3$ to the first column. The fact that $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ are of the form (3.9.1) follows by using the unit columns $\mathbf{e}_i$ to write

$$\mathbf{E}_1 = \mathbf{I}-\mathbf{u}\mathbf{u}^T, \text{ where } \mathbf{u} = \mathbf{e}_1-\mathbf{e}_2, \quad \mathbf{E}_2 = \mathbf{I}-(1-\alpha)\mathbf{e}_2\mathbf{e}_2^T, \quad \text{and} \quad \mathbf{E}_3 = \mathbf{I}+\alpha\mathbf{e}_3\mathbf{e}_1^T.$$

These observations generalize to matrices of arbitrary size.

One of our objectives is to remove the arrows from Gaussian elimination because the inability to do “arrow algebra” limits the theoretical analysis. For example, while it makes sense to add two equations together, there is no meaningful analog for arrows—reducing $\mathbf{A} \to \mathbf{B}$ and $\mathbf{C} \to \mathbf{D}$ by row operations does not guarantee that $\mathbf{A} + \mathbf{C} \to \mathbf{B} + \mathbf{D}$ is possible. The following properties are the mechanisms needed to remove the arrows from elimination processes.

Properties of Elementary Matrices

- When used as a *left-hand* multiplier, an elementary matrix of Type I, II, or III executes the corresponding *row* operation.
- When used as a *right-hand* multiplier, an elementary matrix of Type I, II, or III executes the corresponding *column* operation.

Proof. A proof for Type III operations is given—the other two cases are left to the reader. Using $\mathbf{I} + \alpha \mathbf{e}_j \mathbf{e}_i^T$ as a left-hand multiplier on an arbitrary matrix $\mathbf{A}$ produces

$$\left(\mathbf{I} + \alpha \mathbf{e}_j \mathbf{e}_i^T\right) \mathbf{A} = \mathbf{A} + \alpha \mathbf{e}_j \mathbf{A}_{i*} = \mathbf{A} + \alpha \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{matrix} \\ \\ \longleftarrow \ j^{th} \text{ row} \, . \\ \\ \\ \end{matrix}$$

This is exactly the matrix produced by a Type III row operation in which the i^{th} row of $\mathbf{A}$ is multiplied by α and added to the j^{th} row. When $\mathbf{I} + \alpha \mathbf{e}_j \mathbf{e}_i^T$ is used as a right-hand multiplier on $\mathbf{A}$, the result is

$$\mathbf{A}\left(\mathbf{I} + \alpha \mathbf{e}_j \mathbf{e}_i^T\right) = \mathbf{A} + \alpha \mathbf{A}_{*j} \mathbf{e}_i^T = \mathbf{A} + \alpha \begin{array}{c} \qquad\qquad i^{th} \text{ col} \\ \qquad\qquad \downarrow \\ \begin{pmatrix} 0 & \cdots & a_{1j} & \cdots & 0 \\ 0 & \cdots & a_{2j} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & a_{nj} & \cdots & 0 \end{pmatrix} \end{array}.$$

This is the result of a Type III column operation in which the j^{th} column of $\mathbf{A}$ is multiplied by α and then added to the i^{th} column. ■

Example 3.9.1

The sequence of row operations used to reduce $\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 13 \end{pmatrix}$ to $\mathbf{E_A}$ is indicated below.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 3 & 6 & 13 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \longrightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{Interchange } R_2 \text{ and } R_3} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1 - 4R_2 \\ \\ \\ \end{matrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{E_A}.$$

The reduction can be accomplished by a sequence of left-hand multiplications with the corresponding elementary matrices as shown below.

$$\begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} = \mathbf{E_A}.$$

The product of these elementary matrices is $\mathbf{P} = \begin{pmatrix} 13 & 0 & -4 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$, and you can verify that it is indeed the case that $\mathbf{PA} = \mathbf{E_A}$. Thus the arrows are eliminated by replacing them with a product of elementary matrices.

We are now in a position to understand why nonsingular matrices are precisely those matrices that can be factored as a product of elementary matrices.

Products of Elementary Matrices

- $\mathbf{A}$ is a nonsingular matrix if and only if $\mathbf{A}$ is the product of elementary matrices of Type I, II, or III. (3.9.3)

Proof. If $\mathbf{A}$ is nonsingular, then the Gauss–Jordan technique reduces $\mathbf{A}$ to $\mathbf{I}$ by row operations. If $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_k$ is the sequence of elementary matrices that corresponds to the elementary row operations used, then

$$\mathbf{G}_k \cdots \mathbf{G}_2\mathbf{G}_1\mathbf{A} = \mathbf{I} \text{ or, equivalently, } \mathbf{A} = \mathbf{G}_1^{-1}\mathbf{G}_2^{-1} \cdots \mathbf{G}_k^{-1}.$$

Since the inverse of an elementary matrix is again an elementary matrix of the same type, this proves that $\mathbf{A}$ is the product of elementary matrices of Type I, II, or III. Conversely, if $\mathbf{A} = \mathbf{E}_1\mathbf{E}_2 \cdots \mathbf{E}_k$ is a product of elementary matrices, then $\mathbf{A}$ must be nonsingular because the $\mathbf{E}_i$'s are nonsingular, and a product of nonsingular matrices is also nonsingular. ∎

Equivalence

- Whenever $\mathbf{B}$ can be derived from $\mathbf{A}$ by a combination of elementary row and column operations, we write $\mathbf{A} \sim \mathbf{B}$, and we say that $\mathbf{A}$ and $\mathbf{B}$ are ***equivalent matrices***. Since elementary row and column operations are left-hand and right-hand multiplication by elementary matrices, respectively, and in view of (3.9.3), we can say that
$$\mathbf{A} \sim \mathbf{B} \Longleftrightarrow \mathbf{PAQ} = \mathbf{B} \quad \text{for nonsingular } \mathbf{P} \text{ and } \mathbf{Q}.$$
- Whenever $\mathbf{B}$ can be obtained from $\mathbf{A}$ by performing a sequence of elementary *row* operations only, we write $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$, and we say that $\mathbf{A}$ and $\mathbf{B}$ are ***row equivalent***. In other words,
$$\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \Longleftrightarrow \mathbf{PA} = \mathbf{B} \quad \text{for a nonsingular } \mathbf{P}.$$
- Whenever $\mathbf{B}$ can be obtained from $\mathbf{A}$ by performing a sequence of *column* operations only, we write $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$, and we say that $\mathbf{A}$ and $\mathbf{B}$ are ***column equivalent***. In other words,
$$\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B} \Longleftrightarrow \mathbf{AQ} = \mathbf{B} \quad \text{for a nonsingular } \mathbf{Q}.$$

If it's possible to go from $\mathbf{A}$ to $\mathbf{B}$ by elementary row and column operations, then clearly it's possible to start with $\mathbf{B}$ and get back to $\mathbf{A}$ because elementary operations are reversible—i.e., $\mathbf{PAQ} = \mathbf{B} \implies \mathbf{P}^{-1}\mathbf{BQ}^{-1} = \mathbf{A}$. It therefore makes sense to talk about the equivalence of a pair of matrices without regard to order. In other words, $\mathbf{A} \sim \mathbf{B} \Longleftrightarrow \mathbf{B} \sim \mathbf{A}$. Furthermore, it's not difficult to see that each type of equivalence is ***transitive*** in the sense that

$$\mathbf{A} \sim \mathbf{B} \quad \text{and} \quad \mathbf{B} \sim \mathbf{C} \implies \mathbf{A} \sim \mathbf{C}.$$

In §2.2 it was stated that each matrix $\mathbf{A}$ possesses a unique reduced row echelon form $\mathbf{E_A}$, and we accepted this fact because it is intuitively evident. However, we are now in a position to understand a rigorous proof.

Example 3.9.2

Problem: Prove that $\mathbf{E_A}$ is uniquely determined by $\mathbf{A}$.

Solution: Without loss of generality, we may assume that $\mathbf{A}$ is square—otherwise the appropriate number of zero rows or columns can be adjoined to $\mathbf{A}$ without affecting the results. Suppose that $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{E}_1$ and $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{E}_2$, where $\mathbf{E}_1$ and $\mathbf{E}_2$ are both in reduced row echelon form. Consequently, $\mathbf{E}_1 \overset{\text{row}}{\sim} \mathbf{E}_2$, and hence there is a nonsingular matrix $\mathbf{P}$ such that

$$\mathbf{PE}_1 = \mathbf{E}_2. \tag{3.9.4}$$

Furthermore, by permuting the rows of $\mathbf{E}_1$ and $\mathbf{E}_2$ to force the pivotal 1's to occupy the diagonal positions, we see that

$$\mathbf{E}_1 \overset{\text{row}}{\sim} \mathbf{T}_1 \quad \text{and} \quad \mathbf{E}_2 \overset{\text{row}}{\sim} \mathbf{T}_2, \tag{3.9.5}$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ are upper-triangular matrices in which the basic columns in each $\mathbf{T}_i$ occupy the same positions as the basic columns in $\mathbf{E}_i$. For example, if

$$\mathbf{E} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ then } \mathbf{T} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Each $\mathbf{T}_i$ has the property that $\mathbf{T}_i^2 = \mathbf{T}_i$ because there is a ***permutation matrix*** $\mathbf{Q}_i$ (a product of elementary interchange matrices of Type I) such that

$$\mathbf{Q}_i\mathbf{T}_i\mathbf{Q}_i^T = \begin{pmatrix} \mathbf{I}_{r_i} & \mathbf{J}_i \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \text{ or, equivalently, } \mathbf{T}_i = \mathbf{Q}_i^T \begin{pmatrix} \mathbf{I}_{r_i} & \mathbf{J}_i \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{Q}_i,$$

and $\mathbf{Q}_i^T = \mathbf{Q}_i^{-1}$ (see Exercise 3.9.4) implies $\mathbf{T}_i^2 = \mathbf{T}_i$. It follows from (3.9.5) that $\mathbf{T}_1 \overset{\text{row}}{\sim} \mathbf{T}_2$, so there is a nonsingular matrix $\mathbf{R}$ such that $\mathbf{R}\mathbf{T}_1 = \mathbf{T}_2$. Thus

$$\mathbf{T}_2 = \mathbf{R}\mathbf{T}_1 = \mathbf{R}\mathbf{T}_1\mathbf{T}_1 = \mathbf{T}_2\mathbf{T}_1 \quad \text{and} \quad \mathbf{T}_1 = \mathbf{R}^{-1}\mathbf{T}_2 = \mathbf{R}^{-1}\mathbf{T}_2\mathbf{T}_2 = \mathbf{T}_1\mathbf{T}_2.$$

Because $\mathbf{T}_1$ and $\mathbf{T}_2$ are both upper triangular, $\mathbf{T}_1\mathbf{T}_2$ and $\mathbf{T}_2\mathbf{T}_1$ have the same diagonal entries, and hence $\mathbf{T}_1$ and $\mathbf{T}_2$ have the same diagonal. Therefore, the positions of the basic columns (i.e., the pivotal positions) in $\mathbf{T}_1$ agree with those in $\mathbf{T}_2$, and hence $\mathbf{E}_1$ and $\mathbf{E}_2$ have basic columns in exactly the same positions. This means there is a permutation matrix $\mathbf{Q}$ such that

$$\mathbf{E}_1\mathbf{Q} = \begin{pmatrix} \mathbf{I}_r & \mathbf{J}_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathbf{E}_2\mathbf{Q} = \begin{pmatrix} \mathbf{I}_r & \mathbf{J}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Using (3.9.4) yields $\mathbf{P}\mathbf{E}_1\mathbf{Q} = \mathbf{E}_2\mathbf{Q}$, or

$$\begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{I}_r & \mathbf{J}_1 \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_r & \mathbf{J}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

which in turn implies that $\mathbf{P}_{11} = \mathbf{I}_r$ and $\mathbf{P}_{11}\mathbf{J}_1 = \mathbf{J}_2$. Consequently, $\mathbf{J}_1 = \mathbf{J}_2$, and it follows that $\mathbf{E}_1 = \mathbf{E}_2$.

In passing, notice that the uniqueness of $\mathbf{E_A}$ implies the uniqueness of the pivot positions in any other row echelon form derived from $\mathbf{A}$. If $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{U}_1$ and $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{U}_2$, where $\mathbf{U}_1$ and $\mathbf{U}_2$ are row echelon forms with different pivot positions, then Gauss–Jordan reduction applied to $\mathbf{U}_1$ and $\mathbf{U}_2$ would lead to two different reduced echelon forms, which is impossible.

In §2.2 we observed the fact that the column relationships in a matrix $\mathbf{A}$ are exactly the same as the column relationships in $\mathbf{E_A}$. This observation is a special case of the more general result presented below.

Column and Row Relationships

- If $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$, then linear relationships existing among columns of $\mathbf{A}$ also hold among corresponding columns of $\mathbf{B}$. That is,

$$\mathbf{B}_{*k} = \sum_{j=1}^{n} \alpha_j \mathbf{B}_{*j} \quad \text{if and only if} \quad \mathbf{A}_{*k} = \sum_{j=1}^{n} \alpha_j \mathbf{A}_{*j}. \tag{3.9.6}$$

- In particular, the column relationships in $\mathbf{A}$ and $\mathbf{E_A}$ must be identical, so the nonbasic columns in $\mathbf{A}$ must be linear combinations of the basic columns in $\mathbf{A}$ as described in (2.2.3).
- If $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$, then linear relationships existing among rows of $\mathbf{A}$ must also hold among corresponding rows of $\mathbf{B}$.
- **Summary.** *Row* equivalence preserves *column* relationships, and *column* equivalence preserves *row* relationships.

Proof. If $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$, then $\mathbf{PA} = \mathbf{B}$ for some nonsingular $\mathbf{P}$. Recall from (3.5.5) that the j^{th} column in $\mathbf{B}$ is given by

$$\mathbf{B}_{*j} = (\mathbf{PA})_{*j} = \mathbf{PA}_{*j}.$$

Therefore, if $\mathbf{A}_{*k} = \sum_j \alpha_j \mathbf{A}_{*j}$, then multiplication by $\mathbf{P}$ on the left produces $\mathbf{B}_{*k} = \sum_j \alpha_j \mathbf{B}_{*j}$. Conversely, if $\mathbf{B}_{*k} = \sum_j \alpha_j \mathbf{B}_{*j}$, then multiplication on the left by $\mathbf{P}^{-1}$ produces $\mathbf{A}_{*k} = \sum_j \alpha_j \mathbf{A}_{*j}$. The statement concerning column equivalence follows by considering transposes. ∎

The reduced row echelon form $\mathbf{E_A}$ is as far as we can go in reducing $\mathbf{A}$ by using only row operations. However, if we are allowed to use row operations in conjunction with column operations, then, as described below, the end result of a complete reduction is much simpler.

Rank Normal Form

If $\mathbf{A}$ is an $m \times n$ matrix such that $rank\,(\mathbf{A}) = r$, then

$$\mathbf{A} \sim \mathbf{N}_r = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{3.9.7}$$

$\mathbf{N}_r$ is called the ***rank normal form*** for $\mathbf{A}$, and it is the end product of a complete reduction of $\mathbf{A}$ by using both row and column operations.

Proof. It is always true that $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{E_A}$ so that there is a nonsingular matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{E_A}$. If $rank(\mathbf{A}) = r$, then the basic columns in $\mathbf{E_A}$ are the r unit columns. Apply column interchanges to $\mathbf{E_A}$ so as to move these r unit columns to the far left-hand side. If $\mathbf{Q}_1$ is the product of the elementary matrices corresponding to these column interchanges, then $\mathbf{PAQ}_1$ has the form

$$\mathbf{PAQ}_1 = \mathbf{E_A Q}_1 = \begin{pmatrix} \mathbf{I}_r & \mathbf{J} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Multiplying both sides of this equation on the right by the nonsingular matrix

$$\mathbf{Q}_2 = \begin{pmatrix} \mathbf{I}_r & -\mathbf{J} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \text{ produces } \mathbf{PAQ}_1\mathbf{Q}_2 = \begin{pmatrix} \mathbf{I}_r & \mathbf{J} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{I}_r & -\mathbf{J} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Thus $\mathbf{A} \sim \mathbf{N}_r$. because $\mathbf{P}$ and $\mathbf{Q} = \mathbf{Q}_1\mathbf{Q}_2$ are nonsingular. ∎

Example 3.9.3

Problem: Explain why $rank\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} = rank(\mathbf{A}) + rank(\mathbf{B})$.

Solution: If $rank(\mathbf{A}) = r$ and $rank(\mathbf{B}) = s$, then $\mathbf{A} \sim \mathbf{N}_r$ and $\mathbf{B} \sim \mathbf{N}_s$. Consequently,

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \sim \begin{pmatrix} \mathbf{N}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_s \end{pmatrix} \implies rank\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} = rank\begin{pmatrix} \mathbf{N}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_s \end{pmatrix} = r + s.$$

Given matrices $\mathbf{A}$ and $\mathbf{B}$, how do we decide whether or not $\mathbf{A} \sim \mathbf{B}$, $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$, or $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$? We could use a trial-and-error approach by attempting to reduce $\mathbf{A}$ to $\mathbf{B}$ by elementary operations, but this would be silly because there are easy tests, as described below.

Testing for Equivalence

For $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ the following statements are true.

- $\mathbf{A} \sim \mathbf{B}$ if and only if $rank(\mathbf{A}) = rank(\mathbf{B})$. (3.9.8)
- $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$ if and only if $\mathbf{E_A} = \mathbf{E_B}$. (3.9.9)
- $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$ if and only if $\mathbf{E}_{\mathbf{A}^T} = \mathbf{E}_{\mathbf{B}^T}$. (3.9.10)

Corollary. Multiplication by nonsingular matrices cannot change rank.

Proof. To establish the validity of (3.9.8), observe that $rank\,(\mathbf{A}) = rank\,(\mathbf{B})$ implies $\mathbf{A} \sim \mathbf{N}_r$ and $\mathbf{B} \sim \mathbf{N}_r$. Therefore, $\mathbf{A} \sim \mathbf{N}_r \sim \mathbf{B}$. Conversely, if $\mathbf{A} \sim \mathbf{B}$, where $rank\,(\mathbf{A}) = r$ and $rank\,(\mathbf{B}) = s$, then $\mathbf{A} \sim \mathbf{N}_r$ and $\mathbf{B} \sim \mathbf{N}_s$, and hence $\mathbf{N}_r \sim \mathbf{A} \sim \mathbf{B} \sim \mathbf{N}_s$. Clearly, $\mathbf{N}_r \sim \mathbf{N}_s$ implies $r = s$. To prove (3.9.9), suppose first that $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$. Because $\mathbf{B} \overset{\text{row}}{\sim} \mathbf{E_B}$, it follows that $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{E_B}$. Since a matrix has a uniquely determined reduced echelon form, it must be the case that $\mathbf{E_B} = \mathbf{E_A}$. Conversely, if $\mathbf{E_A} = \mathbf{E_B}$, then

$$\mathbf{A} \overset{\text{row}}{\sim} \mathbf{E_A} = \mathbf{E_B} \overset{\text{row}}{\sim} \mathbf{B} \implies \mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}.$$

The proof of (3.9.10) follows from (3.9.9) by considering transposes because

$$\begin{aligned} \mathbf{A} \overset{\text{col}}{\sim} \mathbf{B} &\iff \mathbf{AQ} = \mathbf{B} \iff (\mathbf{AQ})^T = \mathbf{B}^T \\ &\iff \mathbf{Q}^T\mathbf{A}^T = \mathbf{B}^T \iff \mathbf{A}^T \overset{\text{row}}{\sim} \mathbf{B}^T. \quad \blacksquare \end{aligned}$$

Example 3.9.4

Problem: Are the relationships that exist among the columns in $\mathbf{A}$ the same as the column relationships in $\mathbf{B}$, and are the row relationships in $\mathbf{A}$ the same as the row relationships in $\mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ -4 & -3 & -1 \\ 2 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ 2 & 1 & -1 \end{pmatrix}?$$

Solution: Straightforward computation reveals that

$$\mathbf{E_A} = \mathbf{E_B} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix},$$

and hence $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$. Therefore, the column relationships in $\mathbf{A}$ and $\mathbf{B}$ must be identical, and they must be the same as those in $\mathbf{E_A}$. Examining $\mathbf{E_A}$ reveals that $\mathbf{E}_{*3} = -2\mathbf{E}_{*1} + 3\mathbf{E}_{*2}$, so it must be the case that

$$\mathbf{A}_{*3} = -2\mathbf{A}_{*1} + 3\mathbf{A}_{*2} \quad \text{and} \quad \mathbf{B}_{*3} = -2\mathbf{B}_{*1} + 3\mathbf{B}_{*2}.$$

The row relationships in $\mathbf{A}$ and $\mathbf{B}$ are different because $\mathbf{E}_{\mathbf{A}^T} \neq \mathbf{E}_{\mathbf{B}^T}$.

On the surface, it may not seem plausible that a matrix and its transpose should have the same rank. After all, if $\mathbf{A}$ is 3×100, then $\mathbf{A}$ can have as many as 100 basic columns, but $\mathbf{A}^T$ can have at most three. Nevertheless, we can now demonstrate that $rank\,(\mathbf{A}) = rank\,(\mathbf{A}^T)$.

Transposition and Rank

Transposition does not change the rank—i.e., for all $m \times n$ matrices,

$$rank(\mathbf{A}) = rank\left(\mathbf{A}^T\right) \quad \text{and} \quad rank(\mathbf{A}) = rank(\mathbf{A}^*). \tag{3.9.11}$$

Proof. Let $rank(\mathbf{A}) = r$, and let $\mathbf{P}$ and $\mathbf{Q}$ be nonsingular matrices such that

$$\mathbf{PAQ} = \mathbf{N}_r = \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times n-r} \\ \mathbf{0}_{m-r\times r} & \mathbf{0}_{m-r\times n-r} \end{pmatrix}.$$

Applying the reverse order law for transposition produces $\mathbf{Q}^T\mathbf{A}^T\mathbf{P}^T = \mathbf{N}_r^T$. Since $\mathbf{Q}^T$ and $\mathbf{P}^T$ are nonsingular, it follows that $\mathbf{A}^T \sim \mathbf{N}_r^T$, and therefore

$$rank\left(\mathbf{A}^T\right) = rank\left(\mathbf{N}_r^T\right) = rank\begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r\times m-r} \\ \mathbf{0}_{n-r\times r} & \mathbf{0}_{n-r\times m-r} \end{pmatrix} = r = rank(\mathbf{A}).$$

To prove $rank(\mathbf{A}) = rank(\mathbf{A}^*)$, write $\mathbf{N}_r = \overline{\mathbf{N}_r} = \overline{\mathbf{PAQ}} = \bar{\mathbf{P}}\bar{\mathbf{A}}\bar{\mathbf{Q}}$, and use the fact that the conjugate of a nonsingular matrix is again nonsingular (because $\bar{\mathbf{K}}^{-1} = \overline{\mathbf{K}^{-1}}$) to conclude that $\mathbf{N}_r \sim \bar{\mathbf{A}}$, and hence $rank(\mathbf{A}) = rank\left(\bar{\mathbf{A}}\right)$. It now follows from $rank(\mathbf{A}) = rank\left(\mathbf{A}^T\right)$ that

$$rank(\mathbf{A}^*) = rank\left(\bar{\mathbf{A}}^T\right) = rank\left(\bar{\mathbf{A}}\right) = rank(\mathbf{A}). \quad \blacksquare$$

Exercises for section 3.9

3.9.1. Suppose that $\mathbf{A}$ is an $m \times n$ matrix.

(a) If $[\mathbf{A}|\mathbf{I}_m]$ is row reduced to a matrix $[\mathbf{B}|\mathbf{P}]$, explain why $\mathbf{P}$ must be a nonsingular matrix such that $\mathbf{PA} = \mathbf{B}$.

(b) If $\left[\frac{\mathbf{A}}{\mathbf{I}_n}\right]$ is column reduced to $\left[\frac{\mathbf{C}}{\mathbf{Q}}\right]$, explain why $\mathbf{Q}$ must be a nonsingular matrix such that $\mathbf{AQ} = \mathbf{C}$.

(c) Find a nonsingular matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{E}_{\mathbf{A}}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 6 \end{pmatrix}.$$

(d) Find nonsingular matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{PAQ}$ is in rank normal form.

3.9.2. Consider the two matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 0 & -1 \\ 3 & -1 & 4 & 0 \\ 0 & -8 & 8 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & -6 & 8 & 2 \\ 5 & 1 & 4 & -1 \\ 3 & -9 & 12 & 3 \end{pmatrix}.$$

(a) Are $\mathbf{A}$ and $\mathbf{B}$ equivalent?
(b) Are $\mathbf{A}$ and $\mathbf{B}$ row equivalent?
(c) Are $\mathbf{A}$ and $\mathbf{B}$ column equivalent?

3.9.3. If $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$, explain why the basic columns in $\mathbf{A}$ occupy exactly the same positions as the basic columns in $\mathbf{B}$.

3.9.4. A product of elementary interchange matrices—i.e., elementary matrices of Type I—is called a ***permutation matrix.*** If $\mathbf{P}$ is a permutation matrix, explain why $\mathbf{P}^{-1} = \mathbf{P}^T$.

3.9.5. If $\mathbf{A}_{n\times n}$ is a nonsingular matrix, which (if any) of the following statements are true?

(a) $\mathbf{A} \sim \mathbf{A}^{-1}$. (b) $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{A}^{-1}$. (c) $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{A}^{-1}$.
(d) $\mathbf{A} \sim \mathbf{I}$. (e) $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{I}$. (f) $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{I}$.

3.9.6. Which (if any) of the following statements are true?

(a) $\mathbf{A} \sim \mathbf{B} \implies \mathbf{A}^T \sim \mathbf{B}^T$. (b) $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \implies \mathbf{A}^T \overset{\text{row}}{\sim} \mathbf{B}^T$.
(c) $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \implies \mathbf{A}^T \overset{\text{col}}{\sim} \mathbf{B}^T$. (d) $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \implies \mathbf{A} \sim \mathbf{B}$.
(e) $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B} \implies \mathbf{A} \sim \mathbf{B}$. (f) $\mathbf{A} \sim \mathbf{B} \implies \mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$.

3.9.7. Show that every elementary matrix of Type I can be written as a product of elementary matrices of Types II and III. **Hint:** Recall Exercise 1.2.12 on p. 14.

3.9.8. If $rank\,(\mathbf{A}_{m\times n}) = r$, show that there exist matrices $\mathbf{B}_{m\times r}$ and $\mathbf{C}_{r\times n}$ such that $\mathbf{A} = \mathbf{BC}$, where $rank\,(\mathbf{B}) = rank\,(\mathbf{C}) = r$. Such a factorization is called a ***full-rank factorization.*** **Hint:** Consider the basic columns of $\mathbf{A}$ and the nonzero rows of $\mathbf{E_A}$.

3.9.9. Prove that $rank\,(\mathbf{A}_{m\times n}) = 1$ if and only if there are nonzero columns $\mathbf{u}_{m\times 1}$ and $\mathbf{v}_{n\times 1}$ such that

$$\mathbf{A} = \mathbf{u}\mathbf{v}^T.$$

3.9.10. Prove that if $rank\,(\mathbf{A}_{n\times n}) = 1$, then $\mathbf{A}^2 = \tau\mathbf{A}$, where $\tau = trace\,(\mathbf{A})$.

Solutions for exercises in section 3. 9

3.9.1. (a) If $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_k$ is the sequence of elementary matrices that corresponds to the elementary row operations used in the reduction $[\mathbf{A}|\mathbf{I}] \longrightarrow [\mathbf{B}|\mathbf{P}]$, then

$$\begin{aligned}\mathbf{G}_k \cdots \mathbf{G}_2\mathbf{G}_1[\mathbf{A}|\mathbf{I}] = [\mathbf{B}|\mathbf{P}] \;&\Longrightarrow\; [\mathbf{G}_k \cdots \mathbf{G}_2\mathbf{G}_1\mathbf{A} \mid \mathbf{G}_k \cdots \mathbf{G}_2\mathbf{G}_1\mathbf{I}] = [\mathbf{B}|\mathbf{P}] \\ &\Longrightarrow\; \mathbf{G}_k \cdots \mathbf{G}_2\mathbf{G}_1\mathbf{A} = \mathbf{B} \quad \text{and} \quad \mathbf{G}_k \cdots \mathbf{G}_2\mathbf{G}_1 = \mathbf{P}.\end{aligned}$$

(b) Use the same argument given above, but apply it on the right-hand side.
(c) $[\mathbf{A}|\mathbf{I}] \xrightarrow{\text{Gauss–Jordan}} [\mathbf{E_A}|\mathbf{P}]$ yields

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 \\ 2 & 4 & 6 & 7 & 0 & 1 & 0 \\ 1 & 2 & 3 & 6 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|ccc} 1 & 2 & 3 & 0 & -7 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -5 & 2 & 1 \end{array}\right).$$

Thus $\mathbf{P} = \begin{pmatrix} -7 & 4 & 0 \\ 2 & -1 & 0 \\ -5 & 2 & 1 \end{pmatrix}$ is the product of the elementary matrices corresponding to the operations used in the reduction, and $\mathbf{PA} = \mathbf{E_A}$.
(d) You already have $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{E_A}$. Now find $\mathbf{Q}$ such that $\mathbf{E_A Q} = \mathbf{N}_r$ by column reducing $\mathbf{E_A}$. Proceed using part (b) to accumulate $\mathbf{Q}$.

$$\left[\frac{\mathbf{E_A}}{\mathbf{I}_4}\right] \longrightarrow \left(\begin{array}{cccc} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

3.9.2. (a) Yes—because $rank\,(\mathbf{A}) = rank\,(\mathbf{B})$. (b) Yes—because $\mathbf{E_A} = \mathbf{E_B}$. (c) No—because $\mathbf{E}_{\mathbf{A}^T} \neq \mathbf{E}_{\mathbf{B}^T}$.

3.9.3. The positions of the basic columns in $\mathbf{A}$ correspond to those in $\mathbf{E_A}$. Because $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \Longleftrightarrow \mathbf{E_A} = \mathbf{E_B}$, it follows that the basic columns in $\mathbf{A}$ and $\mathbf{B}$ must be in the same positions.

3.9.4. An elementary interchange matrix (a Type I matrix) has the form $\mathbf{E} = \mathbf{I} - \mathbf{uu}^T$, where $\mathbf{u} = \mathbf{e}_i - \mathbf{e}_j$, and it follows from (3.9.1) that $\mathbf{E} = \mathbf{E}^T = \mathbf{E}^{-1}$. If $\mathbf{P} = \mathbf{E}_1\mathbf{E}_2 \cdots \mathbf{E}_k$ is a product of elementary interchange matrices, then the reverse order laws yield

$$\begin{aligned}\mathbf{P}^{-1} &= (\mathbf{E}_1\mathbf{E}_2 \cdots \mathbf{E}_k)^{-1} = \mathbf{E}_k^{-1} \cdots \mathbf{E}_2^{-1}\mathbf{E}_1^{-1} \\ &= \mathbf{E}_k^T \cdots \mathbf{E}_2^T\mathbf{E}_1^T = (\mathbf{E}_1\mathbf{E}_2 \cdots \mathbf{E}_k)^T = \mathbf{P}^T.\end{aligned}$$

3.9.5. They are all true! $\mathbf{A} \sim \mathbf{I} \sim \mathbf{A}^{-1}$ because $rank\,(\mathbf{A}) = n = rank\,(\mathbf{A}^{-1})$, $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{A}^{-1}$ because $\mathbf{PA} = \mathbf{A}^{-1}$ with $\mathbf{P} = (\mathbf{A}^{-1})^2 = \mathbf{A}^{-2}$, and $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{A}^{-1}$ because $\mathbf{AQ} = \mathbf{A}^{-1}$ with $\mathbf{Q} = \mathbf{A}^{-2}$. The fact that $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{I}$ and $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{I}$ follows since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

3.9.6. (a), (c), (d), and (e) are true.

3.9.7. Rows i and j can be interchanged with the following sequence of Type II and Type III operations—this is Exercise 1.2.12 on p. 14.

$$\begin{array}{ll} R_j \leftarrow R_j + R_i & \text{(replace row } j \text{ by the sum of row } j \text{ and } i) \\ R_i \leftarrow R_i - R_j & \text{(replace row } i \text{ by the difference of row } i \text{ and } j) \\ R_j \leftarrow R_j + R_i & \text{(replace row } j \text{ by the sum of row } j \text{ and } i) \\ R_i \leftarrow -R_i & \text{(replace row } i \text{ by its negative)} \end{array}$$

Translating these to elementary matrices (remembering to build from the right to the left) produces

$$(\mathbf{I} - 2\mathbf{e}_i\mathbf{e}_i^T)(\mathbf{I} + \mathbf{e}_j\mathbf{e}_i^T)(\mathbf{I} - \mathbf{e}_i\mathbf{e}_j^T)(\mathbf{I} + \mathbf{e}_j\mathbf{e}_i^T) = \mathbf{I} - \mathbf{u}\mathbf{u}^T, \quad \text{where} \quad \mathbf{u} = \mathbf{e}_i - \mathbf{e}_j.$$

3.9.8. Let $\mathbf{B}_{m\times r} = [\mathbf{A}_{*b_1}\mathbf{A}_{*b_2} \cdots \mathbf{A}_{*b_r}]$ contain the basic columns of $\mathbf{A}$, and let $\mathbf{C}_{r\times n}$ contain the nonzero rows of $\mathbf{E_A}$. If $\mathbf{A}_{*k}$ is basic—say $\mathbf{A}_{*k} = \mathbf{A}_{*b_j}$ —then $\mathbf{C}_{*k} = \mathbf{e}_j$, and

$$(\mathbf{BC})_{*k} = \mathbf{BC}_{*k} = \mathbf{Be}_j = \mathbf{B}_{*j} = \mathbf{A}_{*b_j} = \mathbf{A}_{*k}.$$

If $\mathbf{A}_{*k}$ is nonbasic, then $\mathbf{C}_{*k}$ is nonbasic and has the form

$$\begin{aligned} \mathbf{C}_{*k} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_j \\ \vdots \\ 0 \end{pmatrix} &= \mu_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \mu_j \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \\ &= \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2 + \cdots + \mu_j\mathbf{e}_j, \end{aligned}$$

where the $\mathbf{e}_i$'s are the basic columns to the left of $\mathbf{C}_{*k}$. Because $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{E_A}$, the relationships that exist among the columns of $\mathbf{A}$ are exactly the same as the relationships that exist among the columns of $\mathbf{E_A}$. In particular,

$$\mathbf{A}_{*k} = \mu_1\mathbf{A}_{*b_1} + \mu_2\mathbf{A}_{*b_2} + \cdots + \mu_j\mathbf{A}_{*b_j},$$

where the $\mathbf{A}_{*b_i}$'s are the basic columns to the left of $\mathbf{A}_{*k}$. Therefore,

$$\begin{aligned} (\mathbf{BC})_{*k} = \mathbf{BC}_{*k} &= \mathbf{B}\,(\mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2 + \cdots + \mu_j\mathbf{e}_j) \\ &= \mu_1\mathbf{B}_{*1} + \mu_2\mathbf{B}_{*2} + \cdots + \mu_j\mathbf{B}_{*j} \\ &= \mu_1\mathbf{A}_{*b_1} + \mu_2\mathbf{A}_{*b_2} + \cdots + \mu_j\mathbf{A}_{*b_j} \\ &= \mathbf{A}_{*k}. \end{aligned}$$

3.9.9. If $\mathbf{A} = \mathbf{u}\mathbf{v}^T$, where $\mathbf{u}_{m\times 1}$ and $\mathbf{v}_{n\times 1}$ are nonzero columns, then

$$\mathbf{u} \overset{\text{row}}{\sim} \mathbf{e}_1 \quad \text{and} \quad \mathbf{v}^T \overset{\text{col}}{\sim} \mathbf{e}_1^T \implies \mathbf{A} = \mathbf{u}\mathbf{v}^T \sim \mathbf{e}_1\mathbf{e}_1^T = \mathbf{N}_1 \implies rank\,(\mathbf{A}) = 1.$$

Conversely, if $rank\,(\mathbf{A}) = 1$, then the existence of $\mathbf{u}$ and $\mathbf{v}$ follows from Exercise 3.9.8. If you do not wish to rely on Exercise 3.9.8, write $\mathbf{PAQ} = \mathbf{N}_1 = \mathbf{e}_1\mathbf{e}_1^T$, where $\mathbf{e}_1$ is $m \times 1$ and $\mathbf{e}_1^T$ is $1 \times n$ so that

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{e}_1\mathbf{e}_1^T\mathbf{Q}^{-1} = \left(\mathbf{P}^{-1}\right)_{*1}\left(\mathbf{Q}^{-1}\right)_{1*} = \mathbf{u}\mathbf{v}^T.$$

3.9.10. Use Exercise 3.9.9 and write

$$\mathbf{A} = \mathbf{u}\mathbf{v}^T \implies \mathbf{A}^2 = \left(\mathbf{u}\mathbf{v}^T\right)\left(\mathbf{u}\mathbf{v}^T\right) = \mathbf{u}\left(\mathbf{v}^T\mathbf{u}\right)\mathbf{v}^T = \tau\mathbf{u}\mathbf{v}^T = \tau\mathbf{A},$$

where $\tau = \mathbf{v}^T\mathbf{u}$. Recall from Example 3.6.5 that $trace\,(\mathbf{AB}) = trace\,(\mathbf{BA})$, and write

$$\tau = trace(\tau) = trace\left(\mathbf{v}^T\mathbf{u}\right) = trace\left(\mathbf{u}\mathbf{v}^T\right) = trace\,(\mathbf{A}).$$

Solutions for exercises in section 3. 10

3.10.1. (a) $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ and $\mathbf{U} = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$ (b) $\mathbf{x}_1 = \begin{pmatrix} 110 \\ -36 \\ 8 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 112 \\ -39 \\ 10 \end{pmatrix}$

(c) $\mathbf{A}^{-1} = \frac{1}{6}\begin{pmatrix} 124 & -40 & 14 \\ -42 & 15 & -6 \\ 10 & -4 & 2 \end{pmatrix}$

3.10.2. (a) The second pivot is zero. (b) $\mathbf{P}$ is the permutation matrix associated with the permutation $\mathbf{p} = (\,2 \quad 4 \quad 1 \quad 3\,)$. $\mathbf{P}$ is constructed by permuting the rows of $\mathbf{I}$ in this manner.

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \\ 2/3 & -1/2 & 1/2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} 3 & 6 & -12 & 3 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 8 & 16 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$

(c) $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$