CHAPTER 4

Vector Spaces

4.1 SPACES AND SUBSPACES

After matrix theory became established toward the end of the nineteenth century, it was realized that many mathematical entities that were considered to be quite different from matrices were in fact quite similar. For example, objects such as points in the plane $\Re^2$, points in 3-space $\Re^3$, polynomials, continuous functions, and differentiable functions (to name only a few) were recognized to satisfy the same additive properties and scalar multiplication properties given in §3.2 for matrices. Rather than studying each topic separately, it was reasoned that it is more efficient and productive to study many topics at one time by studying the common properties that they satisfy. This eventually led to the axiomatic definition of a vector space.

A vector space involves four things—two sets $\mathcal{V}$ and $\mathcal{F}$, and two algebraic operations called vector addition and scalar multiplication.

- $\mathcal{V}$ is a nonempty set of objects called ***vectors***. Although $\mathcal{V}$ can be quite general, we will usually consider $\mathcal{V}$ to be a set of n-tuples or a set of matrices.
- $\mathcal{F}$ is a scalar field—for us $\mathcal{F}$ is either the field $\Re$ of real numbers or the field $\mathcal{C}$ of complex numbers.
- Vector addition (denoted by $\mathbf{x}+\mathbf{y}$) is an operation between elements of $\mathcal{V}$.
- Scalar multiplication (denoted by $\alpha\mathbf{x}$) is an operation between elements of $\mathcal{F}$ and $\mathcal{V}$.

The formal definition of a vector space stipulates how these four things relate to each other. In essence, the requirements are that vector addition and scalar multiplication must obey exactly the same properties given in §3.2 for matrices.

Vector Space Definition

The set $\mathcal{V}$ is called a ***vector space over*** $\mathcal{F}$ when the vector addition and scalar multiplication operations satisfy the following properties.

(A1) $\mathbf{x}+\mathbf{y} \in \mathcal{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$. This is called the ***closure property for vector addition.***

(A2) $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{V}$.

(A3) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ for every $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(A4) There is an element $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{x}+\mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

(A5) For each $\mathbf{x} \in \mathcal{V}$, there is an element $(-\mathbf{x}) \in \mathcal{V}$ such that $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$.

(M1) $\alpha\mathbf{x} \in \mathcal{V}$ for all $\alpha \in \mathcal{F}$ and $\mathbf{x} \in \mathcal{V}$. This is the ***closure property for scalar multiplication.***

(M2) $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for all $\alpha, \beta \in \mathcal{F}$ and every $\mathbf{x} \in \mathcal{V}$.

(M3) $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$ for every $\alpha \in \mathcal{F}$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

(M4) $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x}$ for all $\alpha, \beta \in \mathcal{F}$ and every $\mathbf{x} \in \mathcal{V}$.

(M5) $1\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{V}$.

A theoretical algebraic treatment of the subject would concentrate on the logical consequences of these defining properties, but the objectives in this text are different, so we will not dwell on the axiomatic development.[23] Neverthe-

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[23] The idea of defining a vector space by using a set of abstract axioms was contained in a general theory published in 1844 by Hermann Grassmann (1808–1887), a theologian and philosopher from Stettin, Poland, who was a self-taught mathematician. But Grassmann's work was originally ignored because he tried to construct a highly abstract self-contained theory, independent of the rest of mathematics, containing nonstandard terminology and notation, and he had a tendency to mix mathematics with obscure philosophy. Grassmann published a complete revision of his work in 1862 but with no more success. Only later was it realized that he had formulated the concepts we now refer to as linear dependence, bases, and dimension. The Italian mathematician Giuseppe Peano (1858–1932) was one of the few people who noticed Grassmann's work, and in 1888 Peano published a condensed interpretation of it. In a small chapter at the end, Peano gave an axiomatic definition of a vector space similar to the one above, but this drew little attention outside of a small group in Italy. The current definition is derived from the 1918 work of the German mathematician Hermann Weyl (1885–1955). Even though Weyl's definition is closer to Peano's than to Grassmann's, Weyl did not mention his Italian predecessor, but he did acknowledge Grassmann's "epoch making work." Weyl's success with the idea was due in part to the fact that he thought of vector spaces in terms of geometry, whereas Grassmann and Peano treated them as abstract algebraic structures. As we will see, it's the geometry that's important.

less, it is important to recognize some of the more significant examples and to understand why they are indeed vector spaces.

Example 4.1.1

Because **(A1)–(A5)** are generalized versions of the five additive properties of matrix addition, and **(M1)–(M5)** are generalizations of the five scalar multiplication properties given in §3.2, we can say that the following hold.

- The set $\Re^{m \times n}$ of $m \times n$ real matrices is a vector space over $\Re$.
- The set $\mathcal{C}^{m \times n}$ of $m \times n$ complex matrices is a vector space over $\mathcal{C}$.

Example 4.1.2

The ***real coordinate spaces***

$$\Re^{1 \times n} = \{ (\, x_1 \quad x_2 \quad \cdots \quad x_n \,) \,,\; x_i \in \Re \} \quad \text{and} \quad \Re^{n \times 1} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} ,\; x_i \in \Re \right\}$$

are special cases of the preceding example, and these will be the object of most of our attention. In the context of vector spaces, it usually makes no difference whether a coordinate vector is depicted as a row or as a column. When the row or column distinction is irrelevant, or when it is clear from the context, we will use the common symbol $\Re^n$ to designate a coordinate space. In those cases where it is important to distinguish between rows and columns, we will explicitly write $\Re^{1 \times n}$ or $\Re^{n \times 1}$. Similar remarks hold for complex coordinate spaces.

Although the coordinate spaces will be our primary concern, be aware that there are many other types of mathematical structures that are vector spaces—this was the reason for making an abstract definition at the outset. Listed below are a few examples.

Example 4.1.3

With function addition and scalar multiplication defined by

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x),$$

the following sets are vector spaces over $\Re$:

- The set of functions mapping the interval $[0, 1]$ into $\Re$.
- The set of all real-valued continuous functions defined on $[0, 1]$.
- The set of real-valued functions that are differentiable on $[0, 1]$.
- The set of all polynomials with real coefficients.

Example 4.1.4

Consider the vector space $\Re^2$, and let

$$\mathcal{L} = \{(x, y) \mid y = \alpha x\}$$

be a line through the origin. $\mathcal{L}$ is a subset of $\Re^2$, but $\mathcal{L}$ is a special kind of subset because $\mathcal{L}$ also satisfies the properties **(A1)–(A5)** and **(M1)–(M5)** that define a vector space. This shows that it is possible for one vector space to properly contain other vector spaces.

Subspaces

Let $\mathcal{S}$ be a nonempty subset of a vector space $\mathcal{V}$ over $\mathcal{F}$ (symbolically, $\mathcal{S} \subseteq \mathcal{V}$). If $\mathcal{S}$ is also a vector space over $\mathcal{F}$ using the same addition and scalar multiplication operations, then $\mathcal{S}$ is said to be a ***subspace*** of $\mathcal{V}$. It's not necessary to check all 10 of the defining conditions in order to determine if a subset is also a subspace—only the closure conditions **(A1)** and **(M1)** need to be considered. That is, a nonempty subset $\mathcal{S}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if and only if

(A1) $\quad \mathbf{x}, \mathbf{y} \in \mathcal{S} \implies \mathbf{x} + \mathbf{y} \in \mathcal{S}$

and

(M1) $\quad \mathbf{x} \in \mathcal{S} \implies \alpha\mathbf{x} \in \mathcal{S}$ for all $\alpha \in \mathcal{F}$.

Proof. If $\mathcal{S}$ is a subset of $\mathcal{V}$, then $\mathcal{S}$ automatically inherits all of the vector space properties of $\mathcal{V}$ except **(A1)**, **(A4)**, **(A5)**, and **(M1)**. However, **(A1)** together with **(M1)** implies **(A4)** and **(A5)**. To prove this, observe that **(M1)** implies $(-\mathbf{x}) = (-1)\mathbf{x} \in \mathcal{S}$ for all $\mathbf{x} \in \mathcal{S}$ so that **(A5)** holds. Since $\mathbf{x}$ and $(-\mathbf{x})$ are now both in $\mathcal{S}$, **(A1)** insures that $\mathbf{x} + (-\mathbf{x}) \in \mathcal{S}$, and thus $\mathbf{0} \in \mathcal{S}$. ∎

Example 4.1.5

Given a vector space $\mathcal{V}$, the set $\mathcal{Z} = \{\mathbf{0}\}$ containing only the zero vector is a subspace of $\mathcal{V}$ because **(A1)** and **(M1)** are trivially satisfied. Naturally, this subspace is called the ***trivial subspace.***

Vector addition in $\Re^2$ and $\Re^3$ is easily visualized by using the ***parallelogram law,*** which states that for two vectors $\mathbf{u}$ and $\mathbf{v}$, the sum $\mathbf{u} + \mathbf{v}$ is the vector defined by the diagonal of the parallelogram as shown in Figure 4.1.1.

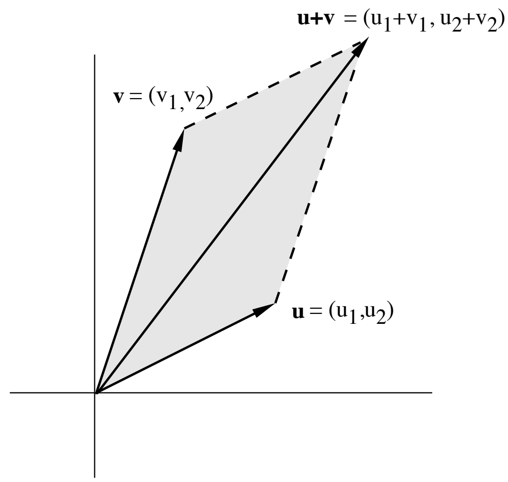

FIGURE 4.1.1

We have already observed that straight lines through the origin in $\Re^2$ are subspaces, but what about straight lines not through the origin? No—they cannot be subspaces because subspaces must contain the zero vector (i.e., they must pass through the origin). What about *curved* lines through the origin—can some of them be subspaces of $\Re^2$? Again the answer is "No!" As depicted in Figure 4.1.2, the parallelogram law indicates why the closure property **(A1)** cannot be satisfied for lines with a curvature because there are points $\mathbf{u}$ and $\mathbf{v}$ on the curve for which $\mathbf{u}+\mathbf{v}$ (the diagonal of the corresponding parallelogram) is not on the curve. Consequently, the only proper subspaces of $\Re^2$ are the trivial subspace and lines through the origin.

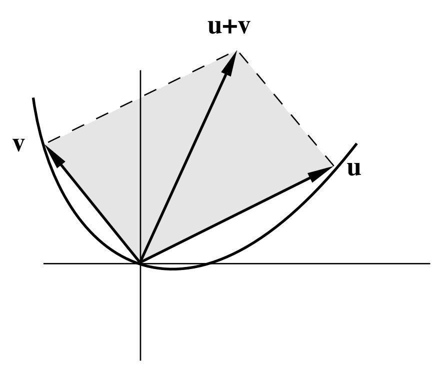

FIGURE 4.1.2

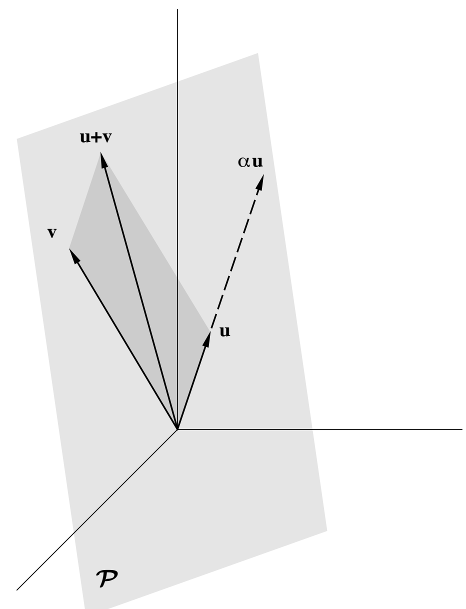

FIGURE 4.1.3

In $\Re^3$, the trivial subspace and lines through the origin are again subspaces, but there is also another one—planes through the origin. If $\mathcal{P}$ is a plane through the origin in $\Re^3$, then, as shown in Figure 4.1.3, the parallelogram law guarantees that the closure property for addition **(A1)** holds—the parallelogram defined by any two vectors in $\mathcal{P}$ is also in $\mathcal{P}$ so that if $\mathbf{u}, \mathbf{v} \in \mathcal{P}$, then $\mathbf{u} + \mathbf{v} \in \mathcal{P}$. The closure property for scalar multiplication **(M1)** holds because multiplying any vector by a scalar merely stretches it, but its angular orientation does not change so that if $\mathbf{u} \in \mathcal{P}$, then $\alpha\mathbf{u} \in \mathcal{P}$ for all scalars α. Lines and surfaces in $\Re^3$ that have curvature cannot be subspaces for essentially the same reason depicted in Figure 4.1.2. So the only proper subspaces of $\Re^3$ are the trivial subspace, lines through the origin, and planes through the origin.

The concept of a subspace now has an obvious interpretation in the visual spaces $\Re^2$ and $\Re^3$—*subspaces are the flat surfaces passing through the origin.*

Flatness

Although we can't use our eyes to see "flatness" in higher dimensions, our minds can conceive it through the notion of a subspace. From now on, think of flat surfaces passing through the origin whenever you encounter the term "subspace."

For a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ from a vector space $\mathcal{V}$, the set of all possible linear combinations of the $\mathbf{v}_i$'s is denoted by

$$span\,(\mathcal{S}) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r \mid \alpha_i \in \mathcal{F}\}\,.$$

Notice that $span\,(\mathcal{S})$ is a subspace of $\mathcal{V}$ because the two closure properties **(A1)** and **(M1)** are satisfied. That is, if $\mathbf{x} = \sum_i \xi_i\mathbf{v}_i$ and $\mathbf{y} = \sum_i \eta_i\mathbf{v}_i$ are two linear combinations from $span\,(\mathcal{S})$, then the sum $\mathbf{x}+\mathbf{y} = \sum_i(\xi_i + \eta_i)\mathbf{v}_i$ is also a linear combination in $span\,(\mathcal{S})$, and for any scalar β, $\beta\mathbf{x} = \sum_i(\beta\xi_i)\mathbf{v}_i$ is also a linear combination in $span\,(\mathcal{S})$.

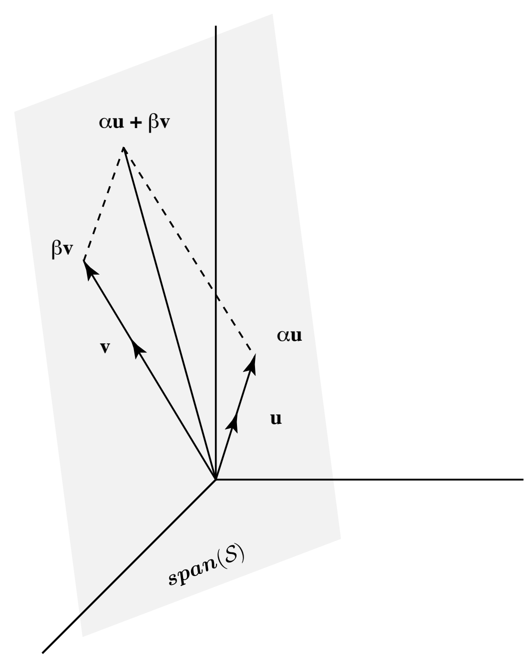

FIGURE 4.1.4

For example, if $\mathbf{u} \neq \mathbf{0}$ is a vector in $\Re^3$, then $span\,\{\mathbf{u}\}$ is the straight line passing through the origin and $\mathbf{u}$. If $\mathcal{S} = \{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u}$ and $\mathbf{v}$ are two nonzero vectors in $\Re^3$ not lying on the same line, then, as shown in Figure 4.1.4, $span\,(\mathcal{S})$ is the plane passing through the origin and the points $\mathbf{u}$ and $\mathbf{v}$. As we will soon see, *all* subspaces of $\Re^n$ are of the type $span\,(\mathcal{S})$, so it is worthwhile to introduce the following terminology.

Spanning Sets

- For a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, the subspace

$$span\,(\mathcal{S}) = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r\}$$

  generated by forming all linear combinations of vectors from $\mathcal{S}$ is called the ***space spanned by*** $\mathcal{S}$.
- If $\mathcal{V}$ is a vector space such that $\mathcal{V} = span\,(\mathcal{S})$, we say $\mathcal{S}$ is a ***spanning set*** for $\mathcal{V}$. In other words, $\mathcal{S}$ ***spans*** $\mathcal{V}$ whenever each vector in $\mathcal{V}$ is a linear combination of vectors from $\mathcal{S}$.

Example 4.1.6

(i) In Figure 4.1.4, $\mathcal{S} = \{\mathbf{u}, \mathbf{v}\}$ is a spanning set for the indicated plane.

(ii) $\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$ spans the line $y = x$ in $\Re^2$.

(iii) The unit vectors $\left\{ \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ span $\Re^3$.

(iv) The unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in $\Re^n$ form a spanning set for $\Re^n$.

(v) The finite set $\left\{1, x, x^2, \ldots, x^n\right\}$ spans the space of all polynomials such that $\deg p(x) \leq n$, and the infinite set $\left\{1, x, x^2, \ldots\right\}$ spans the space of all polynomials.

Example 4.1.7

Problem: For a set of vectors $\mathcal{S} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ from a subspace $\mathcal{V} \subseteq \Re^{m\times 1}$, let $\mathbf{A}$ be the matrix containing the $\mathbf{a}_i$'s as its columns. Explain why $\mathcal{S}$ spans $\mathcal{V}$ if and only if for each $\mathbf{b} \in \mathcal{V}$ there corresponds a column $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$ (i.e., if and only if $\mathbf{Ax} = \mathbf{b}$ is a consistent system for every $\mathbf{b} \in \mathcal{V}$).

Solution: By definition, $\mathcal{S}$ spans $\mathcal{V}$ if and only if for each $\mathbf{b} \in \mathcal{V}$ there exist scalars α_i such that

$$\mathbf{b} = \alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \cdots + \alpha_n\mathbf{a}_n = \big(\mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n\big)\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \mathbf{Ax}.$$

Note: This simple observation often is quite helpful. For example, to test whether or not $\mathcal{S} = \{(1\ \ 1\ \ 1),\ (1\ \ -1\ \ -1),\ (3\ \ 1\ \ 1)\}$ spans $\Re^3$, place these rows as columns in a matrix $\mathbf{A}$, and ask, "Is the system

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

consistent for *every* $\mathbf{b} \in \Re^3$?" Recall from (2.3.4) that $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if $rank[\mathbf{A}|\mathbf{b}] = rank(\mathbf{A})$. In this case, $rank(\mathbf{A}) = 2$, but $rank[\mathbf{A}|\mathbf{b}] = 3$ for some $\mathbf{b}$'s (e.g., $b_1 = 0$, $b_2 = 1$, $b_3 = 0$), so $\mathcal{S}$ doesn't span $\Re^3$. On the other hand, $\mathcal{S}' = \{(1\ \ 1\ \ 1),\ (1\ \ -1\ \ -1),\ (3\ \ 1\ \ 2)\}$ is a spanning set for $\Re^3$ because

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

is nonsingular, so $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b}$ (the solution is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$).

As shown below, it's possible to "add" two subspaces to generate another.

Sum of Subspaces

If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$, then the ***sum*** of $\mathcal{X}$ and $\mathcal{Y}$ is defined to be the set of all possible sums of vectors from $\mathcal{X}$ with vectors from $\mathcal{Y}$. That is,

$$\mathcal{X} + \mathcal{Y} = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in \mathcal{X} \text{ and } \mathbf{y} \in \mathcal{Y}\}.$$

- The sum $\mathcal{X} + \mathcal{Y}$ is again a subspace of $\mathcal{V}$. (4.1.1)
- If $\mathcal{S}_X$, $\mathcal{S}_Y$ span $\mathcal{X}$, $\mathcal{Y}$, then $\mathcal{S}_X \cup \mathcal{S}_Y$ spans $\mathcal{X} + \mathcal{Y}$. (4.1.2)

Proof. To prove (4.1.1), demonstrate that the two closure properties **(A1)** and **(M1)** hold for $\mathcal{S} = \mathcal{X} + \mathcal{Y}$. To show **(A1)** is valid, observe that if $\mathbf{u}, \mathbf{v} \in \mathcal{S}$, then $\mathbf{u} = \mathbf{x}_1 + \mathbf{y}_1$ and $\mathbf{v} = \mathbf{x}_2 + \mathbf{y}_2$, where $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ and $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{Y}$. Because $\mathcal{X}$ and $\mathcal{Y}$ are closed with respect to addition, it follows that $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{X}$ and $\mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{Y}$, and therefore $\mathbf{u} + \mathbf{v} = (\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{y}_1 + \mathbf{y}_2) \in \mathcal{S}$. To verify **(M1)**, observe that $\mathcal{X}$ and $\mathcal{Y}$ are both closed with respect to scalar multiplication so that $\alpha\mathbf{x}_1 \in \mathcal{X}$ and $\alpha\mathbf{y}_1 \in \mathcal{Y}$ for all α, and consequently $\alpha\mathbf{u} = \alpha\mathbf{x}_1 + \alpha\mathbf{y}_1 \in \mathcal{S}$ for all α. To prove (4.1.2), suppose $\mathcal{S}_X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ and $\mathcal{S}_Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_t\}$, and write

$$\begin{aligned} \mathbf{z} \in span\,(\mathcal{S}_X \cup \mathcal{S}_Y) &\Longleftrightarrow \mathbf{z} = \sum_{i=1}^{r} \alpha_i \mathbf{x}_i + \sum_{i=1}^{t} \beta_i \mathbf{y}_i = \mathbf{x} + \mathbf{y} \text{ with } \mathbf{x} \in \mathcal{X},\ \mathbf{y} \in \mathcal{Y} \\ &\Longleftrightarrow \mathbf{z} \in \mathcal{X} + \mathcal{Y}. \quad \blacksquare \end{aligned}$$

Example 4.1.8

If $\mathcal{X} \subseteq \Re^2$ and $\mathcal{Y} \subseteq \Re^2$ are subspaces defined by two different lines through the origin, then $\mathcal{X} + \mathcal{Y} = \Re^2$. This follows from the parallelogram law—sketch a picture for yourself.

Exercises for section 4.1

4.1.1. Determine which of the following subsets of $\Re^n$ are in fact subspaces of $\Re^n$ $(n > 2)$.

(a) $\{\mathbf{x} \mid x_i \geq 0\}$, (b) $\{\mathbf{x} \mid x_1 = 0\}$, (c) $\{\mathbf{x} \mid x_1 x_2 = 0\}$,

(d) $\left\{\mathbf{x} \,\middle|\, \sum_{j=1}^{n} x_j = 0\right\}$, (e) $\left\{\mathbf{x} \,\middle|\, \sum_{j=1}^{n} x_j = 1\right\}$,

(f) $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},$ where $\mathbf{A}_{m\times n} \neq \mathbf{0}$ and $\mathbf{b}_{m\times 1} \neq \mathbf{0}\}$.

4.1.2. Determine which of the following subsets of $\Re^{n\times n}$ are in fact subspaces of $\Re^{n\times n}$.

(a) The symmetric matrices. (b) The diagonal matrices.
(c) The nonsingular matrices. (d) The singular matrices.
(e) The triangular matrices. (f) The upper-triangular matrices.
(g) All matrices that commute with a given matrix $\mathbf{A}$.
(h) All matrices such that $\mathbf{A}^2 = \mathbf{A}$.
(i) All matrices such that $trace\,(\mathbf{A}) = 0$.

4.1.3. If $\mathcal{X}$ is a plane passing through the origin in $\Re^3$ and $\mathcal{Y}$ is the line through the origin that is perpendicular to $\mathcal{X}$, what is $\mathcal{X} + \mathcal{Y}$?

4.1.4. Why must a real or complex nonzero vector space contain an infinite number of vectors?

4.1.5. Sketch a picture in $\Re^3$ of the subspace spanned by each of the following.

(a) $\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}2\\6\\4\end{pmatrix}, \begin{pmatrix}-3\\-9\\-6\end{pmatrix}\right\}$, (b) $\left\{\begin{pmatrix}-4\\0\\0\end{pmatrix}, \begin{pmatrix}0\\5\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$,

(c) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$.

4.1.6. Which of the following are spanning sets for $\Re^3$?

(a) $\{(1 \ \ 1 \ \ 1)\}$ (b) $\{(1 \ \ 0 \ \ 0), (0 \ \ 0 \ \ 1)\}$,
(c) $\{(1 \ \ 0 \ \ 0), (0 \ \ 1 \ \ 0), (0 \ \ 0 \ \ 1), (1 \ \ 1 \ \ 1)\}$,
(d) $\{(1 \ \ 2 \ \ 1), (2 \ \ 0 \ \ -1), (4 \ \ 4 \ \ 1)\}$,
(e) $\{(1 \ \ 2 \ \ 1), (2 \ \ 0 \ \ -1), (4 \ \ 4 \ \ 0)\}$.

4.1.7. For a vector space $\mathcal{V}$, and for $\mathcal{M}, \mathcal{N} \subseteq \mathcal{V}$, explain why $span(\mathcal{M} \cup \mathcal{N}) = span(\mathcal{M}) + span(\mathcal{N})$.

4.1.8. Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of a vector space $\mathcal{V}$.

(a) Prove that the intersection $\mathcal{X} \cap \mathcal{Y}$ is also a subspace of $\mathcal{V}$.
(b) Show that the union $\mathcal{X} \cup \mathcal{Y}$ need not be a subspace of $\mathcal{V}$.

4.1.9. For $\mathbf{A} \in \Re^{m \times n}$ and $\mathcal{S} \subseteq \Re^{n \times 1}$, the set $\mathbf{A}(\mathcal{S}) = \{\mathbf{Ax} \,|\, \mathbf{x} \in \mathcal{S}\}$ contains all possible products of $\mathbf{A}$ with vectors from $\mathcal{S}$. We refer to $\mathbf{A}(\mathcal{S})$ as the set of ***images*** of $\mathcal{S}$ under $\mathbf{A}$.

(a) If $\mathcal{S}$ is a subspace of $\Re^n$, prove $\mathbf{A}(\mathcal{S})$ is a subspace of $\Re^m$.
(b) If $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k$ spans $\mathcal{S}$, show $\mathbf{As}_1, \mathbf{As}_2, \ldots, \mathbf{As}_k$ spans $\mathbf{A}(\mathcal{S})$.

4.1.10. With the usual addition and multiplication, determine whether or not the following sets are vector spaces over the real numbers.

(a) $\Re$, (b) $\mathcal{C}$, (c) The rational numbers.

4.1.11. Let $\mathcal{M} = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r\}$ and $\mathcal{N} = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r, \mathbf{v}\}$ be two sets of vectors from the same vector space. Prove that $span(\mathcal{M}) = span(\mathcal{N})$ if and only if $\mathbf{v} \in span(\mathcal{M})$.

4.1.12. For a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, prove that $span(\mathcal{S})$ is the intersection of all subspaces that contain $\mathcal{S}$. **Hint:** For $\mathcal{M} = \bigcap_{\mathcal{S} \subseteq \mathcal{V}} \mathcal{V}$, prove that $span(\mathcal{S}) \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq span(\mathcal{S})$.

Solutions for Chapter 4

Solutions for exercises in section 4. 1

4.1.1. Only (b) and (d) are subspaces.

4.1.2. (a), (b), (f), (g), and (i) are subspaces.

4.1.3. All of $\Re^3$.

4.1.4. If $\mathbf{v} \in \mathcal{V}$ is a nonzero vector in a space $\mathcal{V}$, then all scalar multiples $\alpha\mathbf{v}$ must also be in $\mathcal{V}$.

4.1.5. (a) A line. (b) The (x,y)-plane. (c) $\Re^3$

4.1.6. Only (c) and (e) span $\Re^3$. To see that (d) does not span $\Re^3$, ask whether or not every vector $(x, y, z) \in \Re^3$ can be written as a linear combination of the vectors in (d). It's convenient to think in terms columns, so rephrase the question by asking if every $\mathbf{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be written as a linear combination of $\left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} \right\}$. That is, for each $\mathbf{b} \in \Re^3$, are there scalars $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{b}$ or, equivalently, is

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{consistent for all} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}?$$

This is a system of the form $\mathbf{Ax} = \mathbf{b}$, and it is consistent for all $\mathbf{b}$ if and only if $rank\,([\mathbf{A}|\mathbf{b}]) = rank\,(\mathbf{A})$ for all $\mathbf{b}$. Since

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & x \\ 2 & 0 & 4 & y \\ 1 & -1 & 1 & z \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & x \\ 0 & -4 & -4 & y - 2x \\ 0 & -3 & -3 & z - x \end{array}\right)$$
$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & x \\ 0 & -4 & -4 & y - 2x \\ 0 & 0 & 0 & (x/2) - (3y/4) + z \end{array}\right),$$

it's clear that there exist $\mathbf{b}$'s (e.g., $\mathbf{b} = (1, 0, 0)^T$) for which $\mathbf{Ax} = \mathbf{b}$ is not consistent, and hence not all $\mathbf{b}$'s are a combination of the $\mathbf{v}_i$'s. Therefore, the $\mathbf{v}_i$'s don't span $\Re^3$.

4.1.7. This follows from (4.1.2).

4.1.8. (a) $\mathbf{u}, \mathbf{v} \in \mathcal{X} \cap \mathcal{Y} \implies \mathbf{u}, \mathbf{v} \in \mathcal{X}$ and $\mathbf{u}, \mathbf{v} \in \mathcal{Y}$. Because $\mathcal{X}$ and $\mathcal{Y}$ are closed with respect to addition, it follows that $\mathbf{u}+\mathbf{v} \in \mathcal{X}$ and $\mathbf{u}+\mathbf{v} \in \mathcal{Y}$, and therefore $\mathbf{u}+\mathbf{v} \in \mathcal{X} \cap \mathcal{Y}$. Because $\mathcal{X}$ and $\mathcal{Y}$ are both closed with respect to scalar multiplication, we have that $\alpha\mathbf{u} \in \mathcal{X}$ and $\alpha\mathbf{u} \in \mathcal{Y}$ for all α, and consequently $\alpha\mathbf{u} \in \mathcal{X} \cap \mathcal{Y}$ for all α.

(b) The union of two different lines through the origin in $\Re^2$ is not a subspace.

4.1.9. (a) **(A1)** holds because $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{A}(\mathcal{S}) \implies \mathbf{x}_1 = \mathbf{A}\mathbf{s}_1$ and $\mathbf{x}_2 = \mathbf{A}\mathbf{s}_2$ for some $\mathbf{s}_1, \mathbf{s}_2 \in \mathcal{S} \implies \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{A}(\mathbf{s}_1 + \mathbf{s}_2)$. Since $\mathcal{S}$ is a subspace, it is closed under vector addition, so $\mathbf{s}_1 + \mathbf{s}_2 \in \mathcal{S}$. Therefore, $\mathbf{x}_1 + \mathbf{x}_2$ is the image of something in $\mathcal{S}$—namely, $\mathbf{s}_1 + \mathbf{s}_2$—and this means that $\mathbf{x}_1 + \mathbf{x}_2 \in \mathbf{A}(\mathcal{S})$. To see that **(M1)** holds, consider $\alpha\mathbf{x}$, where α is an arbitrary scalar and $\mathbf{x} \in \mathbf{A}(\mathcal{S})$. Now, $\mathbf{x} \in \mathbf{A}(\mathcal{S}) \implies \mathbf{x} = \mathbf{A}\mathbf{s}$ for some $\mathbf{s} \in \mathcal{S} \implies \alpha\mathbf{x} = \alpha\mathbf{A}\mathbf{s} = \mathbf{A}(\alpha\mathbf{s})$. Since $\mathcal{S}$ is a subspace, we are guaranteed that $\alpha\mathbf{s} \in \mathcal{S}$, and therefore $\alpha\mathbf{x}$ is the image of something in $\mathcal{S}$. This is what it means to say $\alpha\mathbf{x} \in \mathbf{A}(\mathcal{S})$.

(b) Prove equality by demonstrating that $span\{\mathbf{A}\mathbf{s}_1, \mathbf{A}\mathbf{s}_2, \ldots, \mathbf{A}\mathbf{s}_k\} \subseteq \mathbf{A}(\mathcal{S})$ and $\mathbf{A}(\mathcal{S}) \subseteq span\{\mathbf{A}\mathbf{s}_1, \mathbf{A}\mathbf{s}_2, \ldots, \mathbf{A}\mathbf{s}_k\}$. To show $span\{\mathbf{A}\mathbf{s}_1, \mathbf{A}\mathbf{s}_2, \ldots, \mathbf{A}\mathbf{s}_k\} \subseteq \mathbf{A}(\mathcal{S})$, write

$$\mathbf{x} \in span\{\mathbf{A}\mathbf{s}_1, \mathbf{A}\mathbf{s}_2, \ldots, \mathbf{A}\mathbf{s}_k\} \implies \mathbf{x} = \sum_{i=1}^{k} \alpha_i(\mathbf{A}\mathbf{s}_i) = \mathbf{A}\left(\sum_{i=1}^{k} \alpha_i \mathbf{s}_i\right) \in \mathbf{A}(\mathcal{S}).$$

Inclusion in the reverse direction is established by saying

$$\mathbf{x} \in \mathbf{A}(\mathcal{S}) \implies \mathbf{x} = \mathbf{A}\mathbf{s} \text{ for some } \mathbf{s} \in \mathcal{S} \implies \mathbf{s} = \sum_{i=1}^{k} \beta_i \mathbf{s}_i$$

$$\implies \mathbf{x} = \mathbf{A}\left(\sum_{i=1}^{k} \beta_i \mathbf{s}_i\right) = \sum_{i=1}^{k} \beta_i(\mathbf{A}\mathbf{s}_i) \in span\{\mathbf{A}\mathbf{s}_1, \mathbf{A}\mathbf{s}_2, \ldots, \mathbf{A}\mathbf{s}_k\}.$$

4.1.10. (a) Yes, all of the defining properties are satisfied.

(b) Yes, this is essentially $\Re^2$.

(c) No, it is not closed with respect to scalar multiplication.

4.1.11. If $span(\mathcal{M}) = span(\mathcal{N})$, then every vector in $\mathcal{N}$ must be a linear combination of vectors from $\mathcal{M}$. In particular, $\mathbf{v}$ must be a linear combination of the $\mathbf{m}_i$'s, and hence $\mathbf{v} \in span(\mathcal{M})$. To prove the converse, first notice that $span(\mathcal{M}) \subseteq span(\mathcal{N})$. The desired conclusion will follow if it can be demonstrated that $span(\mathcal{M}) \supseteq span(\mathcal{N})$. The hypothesis that $\mathbf{v} \in span(\mathcal{M})$ guarantees that $\mathbf{v} = \sum_{i=1}^{r} \beta_i \mathbf{m}_i$. If $\mathbf{z} \in span(\mathcal{N})$, then

$$\mathbf{z} = \sum_{i=1}^{r} \alpha_i \mathbf{m}_i + \alpha_{r+1}\mathbf{v} = \sum_{i=1}^{r} \alpha_i \mathbf{m}_i + \alpha_{r+1}\sum_{i=1}^{r} \beta_i \mathbf{m}_i$$

$$= \sum_{i=1}^{r} \Big(\alpha_i + \alpha_{r+1}\beta_i\Big)\mathbf{m}_i,$$

which shows $\mathbf{z} \in span(\mathcal{M})$, and therefore $span(\mathcal{M}) \supseteq span(\mathcal{N})$.

4.1.12. To show $span(\mathcal{S}) \subseteq \mathcal{M}$, observe that $\mathbf{x} \in span(\mathcal{S}) \implies \mathbf{x} = \sum_i \alpha_i \mathbf{v}_i$. If $\mathcal{V}$ is any subspace containing $\mathcal{S}$, then $\sum_i \alpha_i \mathbf{v}_i \in \mathcal{V}$ because $\mathcal{V}$ is closed under addition and scalar multiplication, and therefore $\mathbf{x} \in \mathcal{M}$. The fact that $\mathcal{M} \subseteq span(\mathcal{S})$ follows because if $\mathbf{x} \in \mathcal{M}$, then $\mathbf{x} \in span(\mathcal{S})$ because $span(\mathcal{S})$ is one particular subspace that contains $\mathcal{S}$.

Solutions for exercises in section 4. 2

4.2.1. $R(\mathbf{A}) = span\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}\right\}$, $\quad N\left(\mathbf{A}^T\right) = span\left\{\begin{pmatrix}4\\1\\-2\end{pmatrix}\right\}$,

$$N(\mathbf{A}) = span\left\{\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\-3\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\-4\\0\\1\end{pmatrix}\right\},$$

$$R\left(\mathbf{A}^T\right) = span\left\{\begin{pmatrix}1\\2\\0\\0\\-2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\3\\4\end{pmatrix}\right\}.$$

4.2.2. (a) This is simply a restatement of equation (4.2.3).
(b) $\mathbf{Ax} = \mathbf{b}$ has a unique solution if and only if $rank(\mathbf{A}) = n$ (i.e., there are no free variables—see §2.5), and (4.2.10) says $rank(\mathbf{A}) = n \iff N(\mathbf{A}) = \{\mathbf{0}\}$.

4.2.3. (a) It is consistent because $\mathbf{b} \in R(\mathbf{A})$.
(b) It is nonunique because $N(\mathbf{A}) \neq \{\mathbf{0}\}$—see Exercise 4.2.2.

4.2.4. Yes, because $rank[\mathbf{A}|\mathbf{b}] = rank(\mathbf{A}) = 3 \implies \exists\, \mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$—i.e., $\mathbf{Ax} = \mathbf{b}$ is consistent.

4.2.5. (a) If $R(\mathbf{A}) = \Re^n$, then

$$R(\mathbf{A}) = R(\mathbf{I}_n) \implies \mathbf{A} \overset{\text{col}}{\sim} \mathbf{I}_n \implies rank(\mathbf{A}) = rank(\mathbf{I}_n) = n.$$

(b) $R(\mathbf{A}) = R\left(\mathbf{A}^T\right) = \Re^n$ and $N(\mathbf{A}) = N\left(\mathbf{A}^T\right) = \{\mathbf{0}\}$.

4.2.6. $\mathbf{E_A} \neq \mathbf{E_B}$ means that $R\left(\mathbf{A}^T\right) \neq R\left(\mathbf{B}^T\right)$ and $N(\mathbf{A}) \neq N(\mathbf{B})$. However, $\mathbf{E}_{\mathbf{A}^T} = \mathbf{E}_{\mathbf{B}^T}$ implies that $R(\mathbf{A}) = R(\mathbf{B})$ and $N\left(\mathbf{A}^T\right) = N\left(\mathbf{B}^T\right)$.

4.2.7. Demonstrate that $rank(\mathbf{A}_{n\times n}) = n$ by using (4.2.10). If $\mathbf{x} \in N(\mathbf{A})$, then

$$\begin{aligned}
\mathbf{Ax} = \mathbf{0} &\implies \mathbf{A}_1\mathbf{x} = \mathbf{0} \quad \text{and} \quad \mathbf{A}_2\mathbf{x} = \mathbf{0} \\
&\implies \mathbf{x} \in N(\mathbf{A}_1) = R\left(\mathbf{A}_2^T\right) \implies \exists\, \mathbf{y}^T \text{ such that } \mathbf{x}^T = \mathbf{y}^T\mathbf{A}_2 \\
&\implies \mathbf{x}^T\mathbf{x} = \mathbf{y}^T\mathbf{A}_2\mathbf{x} = \mathbf{0} \implies \sum_i x_i^2 = 0 \implies \mathbf{x} = \mathbf{0}.
\end{aligned}$$