4.2 FOUR FUNDAMENTAL SUBSPACES

The closure properties **(A1)** and **(M1)** on p. 162 that characterize the notion of a subspace have much the same "feel" as the definition of a linear function as stated on p. 89, but there's more to it than just a "similar feel." Subspaces are intimately related to linear functions as explained below.

Subspaces and Linear Functions

For a linear function f mapping $\Re^n$ into $\Re^m$, let $\mathcal{R}(f)$ denote the ***range*** of f. That is, $\mathcal{R}(f) = \{f(\mathbf{x}) \,|\, \mathbf{x} \in \Re^n\} \subseteq \Re^m$ is the set of all "images" as $\mathbf{x}$ varies freely over $\Re^n$.

- The range of every linear function $f : \Re^n \to \Re^m$ is a subspace of $\Re^m$, and every subspace of $\Re^m$ is the range of some linear function.

For this reason, subspaces of $\Re^m$ are sometimes called ***linear spaces***.

Proof. If $f : \Re^n \to \Re^m$ is a linear function, then the range of f is a subspace of $\Re^m$ because the closure properties **(A1)** and **(M1)** are satisfied. Establish **(A1)** by showing that $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(f) \Rightarrow \mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{R}(f)$. If $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}(f)$, then there must be vectors $\mathbf{x}_1, \mathbf{x}_2 \in \Re^n$ such that $\mathbf{y}_1 = f(\mathbf{x}_1)$ and $\mathbf{y}_2 = f(\mathbf{x}_2)$, so it follows from the linearity of f that

$$\mathbf{y}_1 + \mathbf{y}_2 = f(\mathbf{x}_1) + f(\mathbf{x}_2) = f(\mathbf{x}_1 + \mathbf{x}_2) \in \mathcal{R}(f).$$

Similarly, establish **(M1)** by showing that if $\mathbf{y} \in \mathcal{R}(f)$, then $\alpha\mathbf{y} \in \mathcal{R}(f)$ for all scalars α by using the definition of range along with the linearity of f to write

$$\mathbf{y} \in \mathcal{R}(f) \Longrightarrow \mathbf{y} = f(\mathbf{x}) \text{ for some } \mathbf{x} \in \Re^n \Longrightarrow \alpha\mathbf{y} = \alpha f(\mathbf{x}) = f(\alpha\mathbf{x}) \in \mathcal{R}(f).$$

Now prove that every subspace $\mathcal{V}$ of $\Re^m$ is the range of some linear function $f : \Re^n \to \Re^m$. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathcal{V}$ so that

$$\mathcal{V} = \{\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n \,|\, \alpha_i \in \mathcal{R}\}. \tag{4.2.1}$$

Stack the $\mathbf{v}_i$'s as columns in a matrix $\mathbf{A}_{m\times n} = \big(\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \cdots \,|\, \mathbf{v}_n\big)$, and put the α_i's in an $n \times 1$ column $\mathbf{x} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ to write

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n = \big(\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \cdots \,|\, \mathbf{v}_n\big)\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} = \mathbf{A}\mathbf{x}. \tag{4.2.2}$$

The function $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is linear (recall Example 3.6.1, p. 106), and we have that $\mathcal{R}(f) = \{\mathbf{A}\mathbf{x} \,|\, \mathbf{x} \in \Re^{n\times 1}\} = \{\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n \,|\, \alpha_i \in \mathcal{R}\} = \mathcal{V}$. ∎

In particular, this result means that every matrix $\mathbf{A} \in \Re^{m\times n}$ generates a subspace of $\Re^m$ by means of the range of the linear function $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$. Likewise, the transpose[24] of $\mathbf{A} \in \Re^{m\times n}$ defines a subspace of $\Re^n$ by means of the range of $f(\mathbf{y}) = \mathbf{A}^T\mathbf{y}$. These two "range spaces" are two of the four fundamental subspaces defined by a matrix.

Range Spaces

The ***range of a matrix*** $\mathbf{A} \in \Re^{m\times n}$ is defined to be the subspace $R(\mathbf{A})$ of $\Re^m$ that is generated by the range of $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$. That is,

$$R(\mathbf{A}) = \{\mathbf{A}\mathbf{x} \,|\, \mathbf{x} \in \Re^n\} \subseteq \Re^m.$$

Similarly, the range of $\mathbf{A}^T$ is the subspace of $\Re^n$ defined by

$$R\left(\mathbf{A}^T\right) = \{\mathbf{A}^T\mathbf{y} \,|\, \mathbf{y} \in \Re^m\} \subseteq \Re^n.$$

Because $R(\mathbf{A})$ is the set of all "images" of vectors $\mathbf{x} \in \Re^m$ under transformation by $\mathbf{A}$, some people call $R(\mathbf{A})$ the ***image space*** of $\mathbf{A}$.

The observation (4.2.2) that every matrix–vector product $\mathbf{A}\mathbf{x}$ (i.e., every image) is a linear combination of the columns of $\mathbf{A}$ provides a useful characterization of the range spaces. Allowing the components of $\mathbf{x} = (\xi_1, \xi_2, \ldots, \xi_n)^T$ to vary freely and writing

$$\mathbf{A}\mathbf{x} = \Big(\mathbf{A}_{*1} \mid \mathbf{A}_{*2} \mid \cdots \mid \mathbf{A}_{*n}\Big)\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_n\end{pmatrix} = \sum_{j=1}^{n} \xi_j \mathbf{A}_{*j}$$

shows that the set of all images $\mathbf{A}\mathbf{x}$ is the same as the set of all linear combinations of the columns of $\mathbf{A}$. Therefore, $R(\mathbf{A})$ is nothing more than the space spanned by the columns of $\mathbf{A}$. That's why $R(\mathbf{A})$ is often called the ***column space*** of $\mathbf{A}$.

Likewise, $R\left(\mathbf{A}^T\right)$ is the space spanned by the columns of $\mathbf{A}^T$. But the columns of $\mathbf{A}^T$ are just the rows of $\mathbf{A}$ (stacked upright), so $R\left(\mathbf{A}^T\right)$ is simply the space spanned by the rows[25] of $\mathbf{A}$. Consequently, $R\left(\mathbf{A}^T\right)$ is also known as the ***row space*** of $\mathbf{A}$. Below is a summary.

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[24] For ease of exposition, the discussion in this section is in terms of real matrices and real spaces, but all results have complex analogs obtained by replacing $\mathbf{A}^T$ by $\mathbf{A}^*$.

[25] Strictly speaking, the range of $\mathbf{A}^T$ is a set of columns, while the row space of $\mathbf{A}$ is a set of rows. However, no logical difficulties are encountered by considering them to be the same.

Column and Row Spaces

For $\mathbf{A} \in \Re^{m\times n}$, the following statements are true.

- $R(\mathbf{A}) =$ the space spanned by the columns of $\mathbf{A}$ (column space).
- $R(\mathbf{A}^T) =$ the space spanned by the rows of $\mathbf{A}$ (row space).
- $\mathbf{b} \in R(\mathbf{A}) \Longleftrightarrow \mathbf{b} = \mathbf{Ax}$ for some $\mathbf{x}$. (4.2.3)
- $\mathbf{a} \in R(\mathbf{A}^T) \Longleftrightarrow \mathbf{a}^T = \mathbf{y}^T\mathbf{A}$ for some $\mathbf{y}^T$. (4.2.4)

Example 4.2.1

Problem: Describe $R(\mathbf{A})$ and $R(\mathbf{A}^T)$ for $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

Solution: $R(\mathbf{A}) = span\{\mathbf{A}_{*1}, \mathbf{A}_{*2}, \mathbf{A}_{*3}\} = \{\alpha_1\mathbf{A}_{*1} + \alpha_2\mathbf{A}_{*2} + \alpha_3\mathbf{A}_{*3} \mid \alpha_i \in \Re\}$, but since $\mathbf{A}_{*2} = 2\mathbf{A}_{*1}$ and $\mathbf{A}_{*3} = 3\mathbf{A}_{*1}$, it's clear that every linear combination of $\mathbf{A}_{*1}$, $\mathbf{A}_{*2}$, and $\mathbf{A}_{*3}$ reduces to a multiple of $\mathbf{A}_{*1}$, so $R(\mathbf{A}) = span\{\mathbf{A}_{*1}\}$. Geometrically, $R(\mathbf{A})$ is the line in $\Re^2$ through the origin and the point $(1,2)$. Similarly, $R(\mathbf{A}^T) = span\{\mathbf{A}_{1*}, \mathbf{A}_{2*}\} = \{\alpha_1\mathbf{A}_{1*} + \alpha_2\mathbf{A}_{2*} \mid \alpha_1, \alpha_2 \in \Re\}$. But $\mathbf{A}_{2*} = 2\mathbf{A}_{1*}$ implies that every combination of $\mathbf{A}_{1*}$ and $\mathbf{A}_{2*}$ reduces to a multiple of $\mathbf{A}_{1*}$, so $R(\mathbf{A}^T) = span\{\mathbf{A}_{1*}\}$, and this is a line in $\Re^3$ through the origin and the point $(1,2,3)$.

There are times when it is desirable to know whether or not two matrices have the same row space or the same range. The following theorem provides the solution to this problem.

Equal Ranges

For two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same shape:

- $R(\mathbf{A}^T) = R(\mathbf{B}^T)$ if and only if $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$. (4.2.5)
- $R(\mathbf{A}) = R(\mathbf{B})$ if and only if $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$. (4.2.6)

Proof. To prove (4.2.5), first assume $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$ so that there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{B}$. To see that $R(\mathbf{A}^T) = R(\mathbf{B}^T)$, use (4.2.4) to write

$$\begin{aligned} \mathbf{a} \in R(\mathbf{A}^T) &\Longleftrightarrow \mathbf{a}^T = \mathbf{y}^T\mathbf{A} = \mathbf{y}^T\mathbf{P}^{-1}\mathbf{PA} \quad \text{for some } \mathbf{y}^T \\ &\Longleftrightarrow \mathbf{a}^T = \mathbf{z}^T\mathbf{B} \quad \text{for } \mathbf{z}^T = \mathbf{y}^T\mathbf{P}^{-1} \\ &\Longleftrightarrow \mathbf{a} \in R(\mathbf{B}^T). \end{aligned}$$

Conversely, if $R\left(\mathbf{A}^T\right) = R\left(\mathbf{B}^T\right)$, then

$$span\left\{\mathbf{A}_{1*}, \mathbf{A}_{2*}, \ldots, \mathbf{A}_{m*}\right\} = span\left\{\mathbf{B}_{1*}, \mathbf{B}_{2*}, \ldots, \mathbf{B}_{m*}\right\},$$

so each row of $\mathbf{B}$ is a combination of the rows of $\mathbf{A}$, and vice versa. On the basis of this fact, it can be argued that it is possible to reduce $\mathbf{A}$ to $\mathbf{B}$ by using only row operations (the tedious details are omitted), and thus $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$. The proof of (4.2.6) follows by replacing $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{A}^T$ and $\mathbf{B}^T$. ■

Example 4.2.2

Testing Spanning Sets. Two sets $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_s\}$ in $\Re^n$ span the same subspace if and only if the nonzero rows of $\mathbf{E_A}$ agree with the nonzero rows of $\mathbf{E_B}$, where $\mathbf{A}$ and $\mathbf{B}$ are the matrices containing the $\mathbf{a}_i$'s and $\mathbf{b}_i$'s as *rows.* This is a corollary of (4.2.5) because zero rows are irrelevant in considering the row space of a matrix, and we already know from (3.9.9) that $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$ if and only if $\mathbf{E_A} = \mathbf{E_B}$.

Problem: Determine whether or not the following sets span the same subspace:

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 1 \\ 4 \end{pmatrix} \right\}, \qquad \mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

Solution: Place the vectors as *rows* in matrices $\mathbf{A}$ and $\mathbf{B}$, and compute

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{E_A}$$

and

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \mathbf{E_B}.$$

Hence $span\{\mathcal{A}\} = span\{\mathcal{B}\}$ because the nonzero rows in $\mathbf{E_A}$ and $\mathbf{E_B}$ agree.

We already know that the rows of $\mathbf{A}$ span $R\left(\mathbf{A}^T\right)$, and the columns of $\mathbf{A}$ span $R(\mathbf{A})$, but it's often possible to span these spaces with fewer vectors than the full set of rows and columns.

Spanning the Row Space and Range

Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{U}$ be any row echelon form derived from $\mathbf{A}$. Spanning sets for the row and column spaces are as follows:

- The nonzero rows of $\mathbf{U}$ span $R\left(\mathbf{A}^T\right)$. (4.2.7)
- The basic columns in $\mathbf{A}$ span $R(\mathbf{A})$. (4.2.8)

Proof. Statement (4.2.7) is an immediate consequence of (4.2.5). To prove (4.2.8), suppose that the basic columns in $\mathbf{A}$ are in positions $b_1, b_2, \ldots, b_r$, and the nonbasic columns occupy positions $n_1, n_2, \ldots, n_t$, and let $\mathbf{Q}_1$ be the permutation matrix that permutes all of the basic columns in $\mathbf{A}$ to the left-hand side so that $\mathbf{AQ}_1 = (\mathbf{B}_{m\times r} \quad \mathbf{N}_{m\times t})$, where $\mathbf{B}$ contains the basic columns and $\mathbf{N}$ contains the nonbasic columns. Since the nonbasic columns are linear combinations of the basic columns—recall (2.2.3)—we can annihilate the nonbasic columns in $\mathbf{N}$ using elementary column operations. In other words, there is a nonsingular matrix $\mathbf{Q}_2$ such that $(\mathbf{B} \quad \mathbf{N})\mathbf{Q}_2 = (\mathbf{B} \quad \mathbf{0})$. Thus $\mathbf{Q} = \mathbf{Q}_1\mathbf{Q}_2$ is a nonsingular matrix such that $\mathbf{AQ} = \mathbf{AQ}_1\mathbf{Q}_2 = (\mathbf{B} \quad \mathbf{N})\mathbf{Q}_2 = (\mathbf{B} \quad \mathbf{0})$, and hence $\mathbf{A} \overset{\text{col}}{\sim} (\mathbf{B} \quad \mathbf{0})$. The conclusion (4.2.8) now follows from (4.2.6). ■

Example 4.2.3

Problem: Determine spanning sets for $R(\mathbf{A})$ and $R(\mathbf{A}^T)$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}.$$

Solution: Reducing $\mathbf{A}$ to any row echelon form $\mathbf{U}$ provides the solution—the basic columns in $\mathbf{A}$ correspond to the pivotal positions in $\mathbf{U}$, and the nonzero rows of $\mathbf{U}$ span the row space of $\mathbf{A}$. Using $\mathbf{E_A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ produces

$$R(\mathbf{A}) = span\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad R(\mathbf{A}^T) = span\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

So far, only two of the four fundamental subspaces associated with each matrix $\mathbf{A} \in \Re^{m\times n}$ have been discussed, namely, $R(\mathbf{A})$ and $R(\mathbf{A}^T)$. To see where the other two fundamental subspaces come from, consider again a general linear function f mapping $\Re^n$ into $\Re^m$, and focus on $\mathcal{N}(f) = \{\mathbf{x} \mid f(\mathbf{x}) = \mathbf{0}\}$ (the set of vectors that are mapped to $\mathbf{0}$). $\mathcal{N}(f)$ is called the ***nullspace*** of f (some texts call it the ***kernel*** of f), and it's easy to see that $\mathcal{N}(f)$ is a subspace of $\Re^n$ because the closure properties **(A1)** and **(M1)** are satisfied. Indeed, if $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(f)$, then $f(\mathbf{x}_1) = \mathbf{0}$ and $f(\mathbf{x}_2) = \mathbf{0}$, so the linearity of f produces

$$f(\mathbf{x}_1 + \mathbf{x}_2) = f(\mathbf{x}_1) + f(\mathbf{x}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0} \implies \mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{N}(f). \quad \textbf{(A1)}$$

Similarly, if $\alpha \in \Re$, and if $\mathbf{x} \in \mathcal{N}(f)$, then $f(\mathbf{x}) = \mathbf{0}$ and linearity implies

$$f(\alpha\mathbf{x}) = \alpha f(\mathbf{x}) = \alpha\mathbf{0} = \mathbf{0} \implies \alpha\mathbf{x} \in \mathcal{N}(f). \quad \textbf{(M1)}$$

By considering the linear functions $f(\mathbf{x}) = \mathbf{Ax}$ and $g(\mathbf{y}) = \mathbf{A}^T\mathbf{y}$, the other two fundamental subspaces defined by $\mathbf{A} \in \Re^{m\times n}$ are obtained. They are $\mathcal{N}(f) = \{\mathbf{x}_{n\times 1} \mid \mathbf{Ax} = \mathbf{0}\} \subseteq \Re^n$ and $\mathcal{N}(g) = \{\mathbf{y}_{m\times 1} \mid \mathbf{A^T y} = \mathbf{0}\} \subseteq \Re^m$.

Nullspace

- For an $m \times n$ matrix $\mathbf{A}$, the set $N(\mathbf{A}) = \{\mathbf{x}_{n\times 1} \mid \mathbf{A}\mathbf{x} = \mathbf{0}\} \subseteq \Re^n$ is called the ***nullspace*** of $\mathbf{A}$. In other words, $N(\mathbf{A})$ is simply the set of all solutions to the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$.
- The set $N\left(\mathbf{A}^T\right) = \left\{\mathbf{y}_{m\times 1} \mid \mathbf{A}^T\mathbf{y} = \mathbf{0}\right\} \subseteq \Re^m$ is called the ***left-hand nullspace*** of $\mathbf{A}$ because $N\left(\mathbf{A}^T\right)$ is the set of all solutions to the left-hand homogeneous system $\mathbf{y}^T\mathbf{A} = \mathbf{0}^T$.

Example 4.2.4

Problem: Determine a spanning set for $N(\mathbf{A})$, where $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

Solution: $N(\mathbf{A})$ is merely the general solution of $\mathbf{A}\mathbf{x} = \mathbf{0}$, and this is determined by reducing $\mathbf{A}$ to a row echelon form $\mathbf{U}$. As discussed in §2.4, any such $\mathbf{U}$ will suffice, so we will use $\mathbf{E_A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$. Consequently, $x_1 = -2x_2 - 3x_3$, where x_2 and x_3 are free, so the general solution of $\mathbf{A}\mathbf{x} = \mathbf{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

In other words, $N(\mathbf{A})$ is the set of all possible linear combinations of the vectors

$$\mathbf{h}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{h}_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix},$$

and therefore $span\{\mathbf{h}_1, \mathbf{h}_2\} = N(\mathbf{A})$. For this example, $N(\mathbf{A})$ is the plane in $\Re^3$ that passes through the origin and the two points $\mathbf{h}_1$ and $\mathbf{h}_2$.

Example 4.2.4 indicates the general technique for determining a spanning set for $N(\mathbf{A})$. Below is a formal statement of this procedure.

Spanning the Nullspace

To determine a spanning set for $N(\mathbf{A})$, where $rank(\mathbf{A}_{m\times n}) = r$, row reduce $\mathbf{A}$ to a row echelon form $\mathbf{U}$, and solve $\mathbf{Ux} = \mathbf{0}$ for the basic variables in terms of the free variables to produce the general solution of $\mathbf{Ax} = \mathbf{0}$ in the form

$$\mathbf{x} = x_{f_1}\mathbf{h}_1 + x_{f_2}\mathbf{h}_2 + \cdots + x_{f_{n-r}}\mathbf{h}_{n-r}. \tag{4.2.9}$$

By definition, the set $\mathcal{H} = \{\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}\}$ spans $N(\mathbf{A})$. Moreover, it can be proven that $\mathcal{H}$ is unique in the sense that $\mathcal{H}$ is independent of the row echelon form $\mathbf{U}$.

It was established in §2.4 that a homogeneous system $\mathbf{Ax} = \mathbf{0}$ possesses a unique solution (i.e., only the trivial solution $\mathbf{x} = \mathbf{0}$) if and only if the rank of the coefficient matrix equals the number of unknowns. This may now be restated using vector space terminology.

Zero Nullspace

If $\mathbf{A}$ is an $m \times n$ matrix, then

- $N(\mathbf{A}) = \{\mathbf{0}\}$ if and only if $rank(\mathbf{A}) = n$; (4.2.10)
- $N\left(\mathbf{A}^T\right) = \{\mathbf{0}\}$ if and only if $rank(\mathbf{A}) = m$. (4.2.11)

Proof. We already know that the trivial solution $\mathbf{x} = \mathbf{0}$ is the only solution to $\mathbf{Ax} = \mathbf{0}$ if and only if the rank of $\mathbf{A}$ is the number of unknowns, and this is what (4.2.10) says. Similarly, $\mathbf{A}^T\mathbf{y} = \mathbf{0}$ has only the trivial solution $\mathbf{y} = \mathbf{0}$ if and only if $rank\left(\mathbf{A}^T\right) = m$. Recall from (3.9.11) that $rank\left(\mathbf{A}^T\right) = rank(\mathbf{A})$ in order to conclude that (4.2.11) holds. ∎

Finally, let's think about how to determine a spanning set for $N\left(\mathbf{A}^T\right)$. Of course, we can proceed in the same manner as described in Example 4.2.4 by reducing $\mathbf{A}^T$ to a row echelon form to extract the general solution for $\mathbf{A}^T\mathbf{x} = \mathbf{0}$. However, the other three fundamental subspaces are derivable directly from $\mathbf{E_A}$ (or any other row echelon form $\mathbf{U} \overset{\text{row}}{\sim} \mathbf{A}$), so it's rather awkward to have to start from scratch and compute a new echelon form just to get a spanning set for $N\left(\mathbf{A}^T\right)$. It would be better if a single reduction to echelon form could produce all four of the fundamental subspaces. Note that $\mathbf{E}_{\mathbf{A}^T} \neq \mathbf{E}_{\mathbf{A}}^T$, so $\mathbf{E}_{\mathbf{A}}^T$ won't easily lead to $N\left(\mathbf{A}^T\right)$. The following theorem helps resolve this issue.

Left-Hand Nullspace

If $rank(\mathbf{A}_{m\times n}) = r$, and if $\mathbf{PA} = \mathbf{U}$, where $\mathbf{P}$ is nonsingular and $\mathbf{U}$ is in row echelon form, then the last $m - r$ rows in $\mathbf{P}$ span the left-hand nullspace of $\mathbf{A}$. In other words, if $\mathbf{P} = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix}$, where $\mathbf{P}_2$ is $(m-r)\times m$, then

$$N\left(\mathbf{A}^T\right) = R\left(\mathbf{P}_2^T\right). \tag{4.2.12}$$

Proof. If $\mathbf{U} = \begin{pmatrix} \mathbf{C} \\ \mathbf{0} \end{pmatrix}$, where $\mathbf{C}_{r\times n}$, then $\mathbf{PA} = \mathbf{U}$ implies $\mathbf{P}_2\mathbf{A} = \mathbf{0}$, and this says $R\left(\mathbf{P}_2^T\right) \subseteq N\left(\mathbf{A}^T\right)$. To show equality, demonstrate containment in the opposite direction by arguing that every vector in $N\left(\mathbf{A}^T\right)$ must also be in $R\left(\mathbf{P}_2^T\right)$. Suppose $\mathbf{y}^T \in N\left(\mathbf{A}^T\right)$, and let $\mathbf{P}^{-1} = (\mathbf{Q}_1 \quad \mathbf{Q}_2)$ to conclude that

$$\mathbf{0} = \mathbf{y}^T\mathbf{A} = \mathbf{y}^T\mathbf{P}^{-1}\mathbf{U} = \mathbf{y}^T\mathbf{Q}_1\mathbf{C} \implies \mathbf{0} = \mathbf{y}^T\mathbf{Q}_1$$

because $N\left(\mathbf{C}^T\right) = \{\mathbf{0}\}$ by (4.2.11). Now observe that $\mathbf{PP}^{-1} = \mathbf{I} = \mathbf{P}^{-1}\mathbf{P}$ insures $\mathbf{P}_1\mathbf{Q}_1 = \mathbf{I}_r$ and $\mathbf{Q}_1\mathbf{P}_1 = \mathbf{I}_m - \mathbf{Q}_2\mathbf{P}_2$, so

$$\begin{aligned}
\mathbf{0} = \mathbf{y}^T\mathbf{Q}_1 &\implies \mathbf{0} = \mathbf{y}^T\mathbf{Q}_1\mathbf{P}_1 = \mathbf{y}^T\left(\mathbf{I} - \mathbf{Q}_2\mathbf{P}_2\right) \\
&\implies \mathbf{y}^T = \mathbf{y}^T\mathbf{Q}_2\mathbf{P}_2 = \left(\mathbf{y}^T\mathbf{Q}_2\right)\mathbf{P}_2 \\
&\implies \mathbf{y} \in R\left(\mathbf{P}_2^T\right) \implies \mathbf{y}^T \in R\left(\mathbf{P}_2^T\right). \quad \blacksquare
\end{aligned}$$

Example 4.2.5

Problem: Determine a spanning set for $N\left(\mathbf{A}^T\right)$, where $\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}$.

Solution: To find a nonsingular matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{U}$ is in row echelon form, proceed as described in Exercise 3.9.1 and row reduce the augmented matrix $(\mathbf{A} \mid \mathbf{I})$ to $(\mathbf{U} \mid \mathbf{P})$. It must be the case that $\mathbf{PA} = \mathbf{U}$ because $\mathbf{P}$ is the product of the elementary matrices corresponding to the elementary row operations used. Since any row echelon form will suffice, we may use Gauss–Jordan reduction to reduce $\mathbf{A}$ to $\mathbf{E_A}$ as shown below:

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 1 & 3 & 0 & 1 & 0 \\ 3 & 6 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|ccc} 1 & 2 & 0 & 1 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 1 & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & -5/3 & 1 \end{array}\right)$$

$$\mathbf{P} = \begin{pmatrix} -1/3 & 2/3 & 0 \\ 2/3 & -1/3 & 0 \\ 1/3 & -5/3 & 1 \end{pmatrix}, \text{ so (4.2.12) implies } N\left(\mathbf{A}^T\right) = span\left\{\begin{pmatrix} 1/3 \\ -5/3 \\ 1 \end{pmatrix}\right\}.$$

Example 4.2.6

Problem: Suppose $rank\,(\mathbf{A}_{m\times n}) = r$, and let $\mathbf{P} = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix}$ be a nonsingular matrix such that $\mathbf{PA} = \mathbf{U} = \begin{pmatrix} \mathbf{C}_{r\times n} \\ \mathbf{0} \end{pmatrix}$, where $\mathbf{U}$ is in row echelon form. Prove

$$R\,(\mathbf{A}) = N\,(\mathbf{P}_2). \tag{4.2.13}$$

Solution: The strategy is to first prove $R\,(\mathbf{A}) \subseteq N\,(\mathbf{P}_2)$ and then show the reverse inclusion $N\,(\mathbf{P}_2) \subseteq R\,(\mathbf{A})$. The equation $\mathbf{PA} = \mathbf{U}$ implies $\mathbf{P}_2\mathbf{A} = \mathbf{0}$, so all columns of $\mathbf{A}$ are in $N\,(\mathbf{P}_2)$, and thus $R\,(\mathbf{A}) \subseteq N\,(\mathbf{P}_2)$. To show inclusion in the opposite direction, suppose $\mathbf{b} \in N\,(\mathbf{P}_2)$, so that

$$\mathbf{Pb} = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix} \mathbf{b} = \begin{pmatrix} \mathbf{P}_1\mathbf{b} \\ \mathbf{P}_2\mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{d}_{r\times 1} \\ \mathbf{0} \end{pmatrix}.$$

Consequently, $\mathbf{P}\big(\mathbf{A}\,|\,\mathbf{b}\big) = \big(\mathbf{PA} \mid \mathbf{Pb}\big) = \begin{pmatrix} \mathbf{C} & \mathbf{d} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, and this implies

$$rank[\mathbf{A}|\mathbf{b}] = r = rank\,(\mathbf{A}).$$

Recall from (2.3.4) that this means the system $\mathbf{Ax} = \mathbf{b}$ is consistent, and thus $\mathbf{b} \in R\,(\mathbf{A})$ by (4.2.3). Therefore, $N\,(\mathbf{P}_2) \subseteq R\,(\mathbf{A})$, and we may conclude that $N\,(\mathbf{P}_2) = R\,(\mathbf{A})$.

It's often important to know when two matrices have the same nullspace (or left-hand nullspace). Below is one test for determining this.

Equal Nullspaces

For two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same shape:

- $N\,(\mathbf{A}) = N\,(\mathbf{B})$ if and only if $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B}$. (4.2.14)
- $N\left(\mathbf{A}^T\right) = N\left(\mathbf{B}^T\right)$ if and only if $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$. (4.2.15)

Proof. We will prove (4.2.15). If $N\left(\mathbf{A}^T\right) = N\left(\mathbf{B}^T\right)$, then (4.2.12) guarantees $R\left(\mathbf{P}_2^T\right) = N\left(\mathbf{B}^T\right)$, and hence $\mathbf{P}_2\mathbf{B} = \mathbf{0}$. But this means the columns of $\mathbf{B}$ are in $N\,(\mathbf{P}_2)$. That is, $R\,(\mathbf{B}) \subseteq N\,(\mathbf{P}_2) = R\,(\mathbf{A})$ by using (4.2.13). If $\mathbf{A}$ is replaced by $\mathbf{B}$ in the preceding argument—and in (4.2.13)— the result is that $R\,(\mathbf{A}) \subseteq R\,(\mathbf{B})$, and consequently we may conclude that $R\,(\mathbf{A}) = R\,(\mathbf{B})$. The desired conclusion (4.2.15) follows from (4.2.6). Statement (4.2.14) now follows by replacing $\mathbf{A}$ and $\mathbf{B}$ by $\mathbf{A}^T$ and $\mathbf{B}^T$ in (4.2.15). ∎

Summary

The four fundamental subspaces associated with $\mathbf{A}_{m\times n}$ are as follows.

- The range or column space: $R(\mathbf{A}) = \{\mathbf{Ax}\} \subseteq \Re^m$.
- The row space or left-hand range: $R(\mathbf{A}^T) = \{\mathbf{A}^T\mathbf{y}\} \subseteq \Re^n$.
- The nullspace: $N(\mathbf{A}) = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{0}\} \subseteq \Re^n$.
- The left-hand nullspace: $N(\mathbf{A}^T) = \{\mathbf{y} \mid \mathbf{A}^T\mathbf{y} = \mathbf{0}\} \subseteq \Re^m$.

Let $\mathbf{P}$ be a nonsingular matrix such that $\mathbf{PA} = \mathbf{U}$, where $\mathbf{U}$ is in row echelon form, and suppose $rank(\mathbf{A}) = r$.

- Spanning set for $R(\mathbf{A})$ = the basic columns in $\mathbf{A}$.
- Spanning set for $R(\mathbf{A}^T)$ = the nonzero rows in $\mathbf{U}$.
- Spanning set for $N(\mathbf{A})$ =the $\mathbf{h}_i$'s in the general solution of $\mathbf{Ax} = \mathbf{0}$.
- Spanning set for $N(\mathbf{A}^T)$ = the last $m-r$ rows of $\mathbf{P}$.

If $\mathbf{A}$ and $\mathbf{B}$ have the same shape, then

- $\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \Longleftrightarrow N(\mathbf{A}) = N(\mathbf{B}) \Longleftrightarrow R(\mathbf{A}^T) = R(\mathbf{B}^T)$.
- $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B} \Longleftrightarrow R(\mathbf{A}) = R(\mathbf{B}) \Longleftrightarrow N(\mathbf{A}^T) = N(\mathbf{B}^T)$.

Exercises for section 4.2

4.2.1. Determine spanning sets for each of the four fundamental subspaces associated with

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix}.$$

4.2.2. Consider a linear system of equations $\mathbf{A}_{m\times n}\mathbf{x} = \mathbf{b}$.

(a) Explain why $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if $\mathbf{b} \in R(\mathbf{A})$.

(b) Explain why a consistent system $\mathbf{Ax} = \mathbf{b}$ has a unique solution if and only if $N(\mathbf{A}) = \{\mathbf{0}\}$.

4.2.3. Suppose that $\mathbf{A}$ is a 3×3 matrix such that

$$\mathcal{R} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{N} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

span $R(\mathbf{A})$ and $N(\mathbf{A})$, respectively, and consider a linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 1 \\ -7 \\ 0 \end{pmatrix}$.

(a) Explain why $\mathbf{Ax} = \mathbf{b}$ must be consistent.
(b) Explain why $\mathbf{Ax} = \mathbf{b}$ cannot have a unique solution.

4.2.4. If $\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 & -2 & 1 \\ -1 & 0 & 3 & -4 & 2 \\ -1 & 0 & 3 & -5 & 3 \\ -1 & 0 & 3 & -6 & 4 \\ -1 & 0 & 3 & -6 & 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ -5 \\ -6 \\ -7 \\ -7 \end{pmatrix}$, is $\mathbf{b} \in R(\mathbf{A})$?

4.2.5. Suppose that $\mathbf{A}$ is an $n \times n$ matrix.

(a) If $R(\mathbf{A}) = \Re^n$, explain why $\mathbf{A}$ must be nonsingular.
(b) If $\mathbf{A}$ is nonsingular, describe its four fundamental subspaces.

4.2.6. Consider the matrices $\mathbf{A} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -4 & 4 \\ 4 & -8 & 6 \\ 0 & -4 & 5 \end{pmatrix}$.

(a) Do $\mathbf{A}$ and $\mathbf{B}$ have the same row space?
(b) Do $\mathbf{A}$ and $\mathbf{B}$ have the same column space?
(c) Do $\mathbf{A}$ and $\mathbf{B}$ have the same nullspace?
(d) Do $\mathbf{A}$ and $\mathbf{B}$ have the same left-hand nullspace?

4.2.7. If $\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix}$ is a square matrix such that $N(\mathbf{A}_1) = R\left(\mathbf{A}_2^T\right)$, prove that $\mathbf{A}$ must be nonsingular.

4.2.8. Consider a linear system of equations $\mathbf{Ax} = \mathbf{b}$ for which $\mathbf{y}^T\mathbf{b} = 0$ for every $\mathbf{y} \in N\left(\mathbf{A}^T\right)$. Explain why this means the system must be consistent.

4.2.9. For matrices $\mathbf{A}_{m \times n}$ and $\mathbf{B}_{m \times p}$, prove that

$$R(\mathbf{A} \mid \mathbf{B}) = R(\mathbf{A}) + R(\mathbf{B}).$$

4.2.10. Let $\mathbf{p}$ be one particular solution of a linear system $\mathbf{Ax} = \mathbf{b}$.

(a) Explain the significance of the set

$$\mathbf{p} + N(\mathbf{A}) = \{\mathbf{p} + \mathbf{h} \mid \mathbf{h} \in N(\mathbf{A})\}.$$

(b) If $rank(\mathbf{A}_{3\times 3}) = 1$, sketch a picture of $\mathbf{p} + N(\mathbf{A})$ in $\Re^3$.

(c) Repeat part (b) for the case when $rank(\mathbf{A}_{3\times 3}) = 2$.

4.2.11. Suppose that $\mathbf{Ax} = \mathbf{b}$ is a consistent system of linear equations, and let $\mathbf{a} \in R(\mathbf{A}^T)$. Prove that the inner product $\mathbf{a}^T\mathbf{x}$ is constant for all solutions to $\mathbf{Ax} = \mathbf{b}$.

4.2.12. For matrices such that the product $\mathbf{AB}$ is defined, explain why each of the following statements is true.

(a) $R(\mathbf{AB}) \subseteq R(\mathbf{A})$.

(b) $N(\mathbf{AB}) \supseteq N(\mathbf{B})$.

4.2.13. Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a spanning set for $R(\mathbf{B})$. Prove that $\mathbf{A}(\mathcal{B}) = \{\mathbf{Ab}_1, \mathbf{Ab}_2, \ldots, \mathbf{Ab}_n\}$ is a spanning set for $R(\mathbf{AB})$.

which shows $\mathbf{z} \in span(\mathcal{M})$, and therefore $span(\mathcal{M}) \supseteq span(\mathcal{N})$.

4.1.12. To show $span(\mathcal{S}) \subseteq \mathcal{M}$, observe that $\mathbf{x} \in span(\mathcal{S}) \implies \mathbf{x} = \sum_i \alpha_i \mathbf{v}_i$. If $\mathcal{V}$ is any subspace containing $\mathcal{S}$, then $\sum_i \alpha_i \mathbf{v}_i \in \mathcal{V}$ because $\mathcal{V}$ is closed under addition and scalar multiplication, and therefore $\mathbf{x} \in \mathcal{M}$. The fact that $\mathcal{M} \subseteq span(\mathcal{S})$ follows because if $\mathbf{x} \in \mathcal{M}$, then $\mathbf{x} \in span(\mathcal{S})$ because $span(\mathcal{S})$ is one particular subspace that contains $\mathcal{S}$.

Solutions for exercises in section 4. 2

4.2.1. $R(\mathbf{A}) = span\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}\right\}, \quad N\left(\mathbf{A}^T\right) = span\left\{\begin{pmatrix}4\\1\\-2\end{pmatrix}\right\},$

$$N(\mathbf{A}) = span\left\{\begin{pmatrix}-2\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\-3\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\-4\\0\\1\end{pmatrix}\right\},$$

$$R\left(\mathbf{A}^T\right) = span\left\{\begin{pmatrix}1\\2\\0\\0\\-2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\3\\4\end{pmatrix}\right\}.$$

4.2.2. (a) This is simply a restatement of equation (4.2.3).
(b) $\mathbf{Ax} = \mathbf{b}$ has a unique solution if and only if $rank(\mathbf{A}) = n$ (i.e., there are no free variables—see §2.5), and (4.2.10) says $rank(\mathbf{A}) = n \iff N(\mathbf{A}) = \{\mathbf{0}\}$.

4.2.3. (a) It is consistent because $\mathbf{b} \in R(\mathbf{A})$.
(b) It is nonunique because $N(\mathbf{A}) \neq \{\mathbf{0}\}$ —see Exercise 4.2.2.

4.2.4. Yes, because $rank[\mathbf{A}|\mathbf{b}] = rank(\mathbf{A}) = 3 \implies \exists\, \mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$—i.e., $\mathbf{Ax} = \mathbf{b}$ is consistent.

4.2.5. (a) If $R(\mathbf{A}) = \Re^n$, then

$$R(\mathbf{A}) = R(\mathbf{I}_n) \implies \mathbf{A} \overset{\text{col}}{\sim} \mathbf{I}_n \implies rank(\mathbf{A}) = rank(\mathbf{I}_n) = n.$$

(b) $R(\mathbf{A}) = R\left(\mathbf{A}^T\right) = \Re^n$ and $N(\mathbf{A}) = N\left(\mathbf{A}^T\right) = \{\mathbf{0}\}$.

4.2.6. $\mathbf{E_A} \neq \mathbf{E_B}$ means that $R\left(\mathbf{A}^T\right) \neq R\left(\mathbf{B}^T\right)$ and $N(\mathbf{A}) \neq N(\mathbf{B})$. However, $\mathbf{E}_{\mathbf{A}^T} = \mathbf{E}_{\mathbf{B}^T}$ implies that $R(\mathbf{A}) = R(\mathbf{B})$ and $N\left(\mathbf{A}^T\right) = N\left(\mathbf{B}^T\right)$.

4.2.7. Demonstrate that $rank(\mathbf{A}_{n\times n}) = n$ by using (4.2.10). If $\mathbf{x} \in N(\mathbf{A})$, then

$$\begin{aligned}\mathbf{Ax} = \mathbf{0} &\implies \mathbf{A}_1\mathbf{x} = \mathbf{0} \quad \text{and} \quad \mathbf{A}_2\mathbf{x} = \mathbf{0}\\ &\implies \mathbf{x} \in N(\mathbf{A}_1) = R\left(\mathbf{A}_2^T\right) \implies \exists\, \mathbf{y}^T \text{ such that } \mathbf{x}^T = \mathbf{y}^T\mathbf{A}_2\\ &\implies \mathbf{x}^T\mathbf{x} = \mathbf{y}^T\mathbf{A}_2\mathbf{x} = \mathbf{0} \implies \sum_i x_i^2 = 0 \implies \mathbf{x} = \mathbf{0}.\end{aligned}$$

4.2.8. $\mathbf{y}^T\mathbf{b} = 0 \ \forall \ \mathbf{y} \in N\left(\mathbf{A}^T\right) = R\left(\mathbf{P}_2^T\right) \implies \mathbf{P}_2\mathbf{b} = 0 \implies \mathbf{b} \in N(\mathbf{P}_2) = R(\mathbf{A})$

4.2.9. $\mathbf{x} \in R\big(\mathbf{A} \mid \mathbf{B}\big) \iff \exists \ \mathbf{y}$ such that $\mathbf{x} = \big(\mathbf{A} \mid \mathbf{B}\big)\mathbf{y} = \big(\mathbf{A} \mid \mathbf{B}\big)\begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{pmatrix} = \mathbf{A}\mathbf{y}_1 + \mathbf{B}\mathbf{y}_2 \iff \mathbf{x} \in R(\mathbf{A}) + R(\mathbf{B})$

4.2.10. (a) $\mathbf{p} + N(\mathbf{A})$ is the set of all possible solutions to $\mathbf{Ax} = \mathbf{b}$. Recall from (2.5.7) that the general solution of a nonhomogeneous equation is a particular solution plus the general solution of the homogeneous equation $\mathbf{Ax} = \mathbf{0}$. The general solution of the homogeneous equation is simply a way of describing all possible solutions of $\mathbf{Ax} = \mathbf{0}$, which is $N(\mathbf{A})$.

(b) $rank(\mathbf{A}_{3\times 3}) = 1$ means that $N(\mathbf{A})$ is spanned by two vectors, and hence $N(\mathbf{A})$ is a plane through the origin. From the parallelogram law, $\mathbf{p} + N(\mathbf{A})$ is a plane parallel to $N(\mathbf{A})$ passing through the point defined by $\mathbf{p}$.

(c) This time $N(\mathbf{A})$ is spanned by a single vector, and $\mathbf{p} + N(\mathbf{A})$ is a line parallel to $N(\mathbf{A})$ passing through the point defined by $\mathbf{p}$.

4.2.11. $\mathbf{a} \in R\left(\mathbf{A}^T\right) \iff \exists \ \mathbf{y}$ such that $\mathbf{a}^T = \mathbf{y}^T\mathbf{A}$. If $\mathbf{Ax} = \mathbf{b}$, then

$$\mathbf{a}^T\mathbf{x} = \mathbf{y}^T\mathbf{Ax} = \mathbf{y}^T\mathbf{b},$$

which is independent of $\mathbf{x}$.

4.2.12. (a) $\mathbf{b} \in R(\mathbf{AB}) \implies \exists \ \mathbf{x}$ such that $\mathbf{b} = \mathbf{ABx} = \mathbf{A}(\mathbf{Bx}) \implies \mathbf{b} \in R(\mathbf{A})$ because $\mathbf{b}$ is the image of $\mathbf{Bx}$.

(b) $\mathbf{x} \in N(\mathbf{B}) \implies \mathbf{Bx} = \mathbf{0} \implies \mathbf{ABx} = \mathbf{0} \implies \mathbf{x} \in N(\mathbf{AB})$.

4.2.13. Given any $\mathbf{z} \in R(\mathbf{AB})$, the object is to show that $\mathbf{z}$ can be written as some linear combination of the $\mathbf{Ab}_i$'s. Argue as follows. $\mathbf{z} \in R(\mathbf{AB}) \implies \mathbf{z} = \mathbf{ABy}$ for some $\mathbf{y}$. But it is always true that $\mathbf{By} \in R(\mathbf{B})$, so

$$\mathbf{By} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \cdots + \alpha_n\mathbf{b}_n,$$

and therefore $\mathbf{z} = \mathbf{ABy} = \alpha_1\mathbf{Ab}_1 + \alpha_2\mathbf{Ab}_2 + \cdots + \alpha_n\mathbf{Ab}_n$.

Solutions for exercises in section 4. 3

4.3.1. (a) and (b) are linearly dependent—all others are linearly independent. To write one vector as a combination of others in a dependent set, place the vectors as columns in $\mathbf{A}$ and find $\mathbf{E_A}$. This reveals the dependence relationships among columns of $\mathbf{A}$.

4.3.2. (a) According to (4.3.12), the basic columns in $\mathbf{A}$ always constitute one maximal linearly independent subset.

(b) Ten—5 sets using two vectors, 4 sets using one vector, and the empty set.

4.3.3. $rank(\mathbf{H}) \leq 3$, and according to (4.3.11), $rank(\mathbf{H})$ is the maximal number of independent rows in $\mathbf{H}$.