4.3 LINEAR INDEPENDENCE

For a given set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ there may or may not exist dependency relationships in the sense that it may or may not be possible to express one vector as a linear combination of the others. For example, in the set

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 9 \\ -3 \\ 4 \end{pmatrix} \right\},$$

the third vector is a linear combination of the first two—i.e., $\mathbf{v}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2$. Such a dependency always can be expressed in terms of a homogeneous equation by writing

$$3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}.$$

On the other hand, it is evident that there are no dependency relationships in the set

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

because no vector can be expressed as a combination of the others. Another way to say this is to state that there are no solutions for α_1, α_2, and α_3 in the homogeneous equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$$

other than the trivial solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$. These observations are the basis for the following definitions.

Linear Independence

A set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be a ***linearly independent set*** whenever the only solution for the scalars α_i in the homogeneous equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0} \qquad (4.3.1)$$

is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Whenever there is a nontrivial solution for the α's (i.e., at least one $\alpha_i \neq 0$) in (4.3.1), the set $\mathcal{S}$ is said to be a ***linearly dependent set.*** In other words, linearly independent sets are those that contain no dependency relationships, and linearly dependent sets are those in which at least one vector is a combination of the others. We will agree that the empty set is always linearly independent.

It is important to realize that the concepts of linear independence and dependence are defined only for sets—individual vectors are neither linearly independent nor dependent. For example consider the following sets:

$$\mathcal{S}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{S}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{S}_3 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

It should be clear that $\mathcal{S}_1$ and $\mathcal{S}_2$ are linearly independent sets while $\mathcal{S}_3$ is linearly dependent. This shows that individual vectors can simultaneously belong to linearly independent sets as well as linearly dependent sets. Consequently, it makes no sense to speak of "linearly independent vectors" or "linearly dependent vectors."

Example 4.3.1

Problem: Determine whether or not the set

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \right\}$$

is linearly independent.

Solution: Simply determine whether or not there exists a nontrivial solution for the α's in the homogeneous equation

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or, equivalently, if there is a nontrivial solution to the homogeneous system

$$\begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

If $\mathbf{A} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{pmatrix}$, then $\mathbf{E_A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, and therefore there exist nontrivial solutions. Consequently, $\mathcal{S}$ is a linearly dependent set. Notice that one particular dependence relationship in $\mathcal{S}$ is revealed by $\mathbf{E_A}$ because it guarantees that $\mathbf{A}_{*3} = 3\mathbf{A}_{*1} + 2\mathbf{A}_{*2}$. This example indicates why the question of whether or not a subset of $\Re^m$ is linearly independent is really a question about whether or not the nullspace of an associated matrix is trivial. The following is a more formal statement of this fact.

Linear Independence and Matrices

Let $\mathbf{A}$ be an $m \times n$ matrix.

- Each of the following statements is equivalent to saying that the columns of $\mathbf{A}$ form a linearly independent set.
  - ▷ $N(\mathbf{A}) = \{\mathbf{0}\}$. (4.3.2)
  - ▷ $rank(\mathbf{A}) = n$. (4.3.3)
- Each of the following statements is equivalent to saying that the rows of $\mathbf{A}$ form a linearly independent set.
  - ▷ $N\left(\mathbf{A}^T\right) = \{\mathbf{0}\}$. (4.3.4)
  - ▷ $rank(\mathbf{A}) = m$. (4.3.5)
- When $\mathbf{A}$ is a square matrix, each of the following statements is equivalent to saying that $\mathbf{A}$ is nonsingular.
  - ▷ The columns of $\mathbf{A}$ form a linearly independent set. (4.3.6)
  - ▷ The rows of $\mathbf{A}$ form a linearly independent set. (4.3.7)

Proof. By definition, the columns of $\mathbf{A}$ are a linearly independent set when the only set of α's satisfying the homogeneous equation

$$\mathbf{0} = \alpha_1\mathbf{A}_{*1} + \alpha_2\mathbf{A}_{*2} + \cdots + \alpha_n\mathbf{A}_{*n} = \left(\mathbf{A}_{*1} \,|\, \mathbf{A}_{*2} \,|\, \cdots \,|\, \mathbf{A}_{*n}\right)\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}$$

is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, which is equivalent to saying $N(\mathbf{A}) = \{\mathbf{0}\}$. The fact that $N(\mathbf{A}) = \{\mathbf{0}\}$ is equivalent to $rank(\mathbf{A}) = n$ was demonstrated in (4.2.10). Statements (4.3.4) and (4.3.5) follow by replacing $\mathbf{A}$ by $\mathbf{A}^T$ in (4.3.2) and (4.3.3) and by using the fact that $rank(\mathbf{A}) = rank\left(\mathbf{A}^T\right)$. Statements (4.3.6) and (4.3.7) are simply special cases of (4.3.3) and (4.3.5). ∎

Example 4.3.2

Any set $\{\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_n}\}$ consisting of distinct unit vectors is a linearly independent set because $rank\left(\mathbf{e}_{i_1} \,|\, \mathbf{e}_{i_2} \,|\, \cdots \,|\, \mathbf{e}_{i_n}\right) = n$. For example, the set of unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}$ in $\Re^4$ is linearly independent because $rank\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&1\end{pmatrix} = 3$.

Example 4.3.3

Diagonal Dominance. A matrix $\mathbf{A}_{n\times n}$ is said to be ***diagonally dominant*** whenever

$$|a_{ii}| > \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}| \quad \text{for each } i = 1, 2, \ldots, n.$$

That is, the magnitude of each diagonal entry exceeds the sum of the magnitudes of the off-diagonal entries in the corresponding row. Diagonally dominant matrices occur naturally in a wide variety of practical applications, and when solving a diagonally dominant system by Gaussian elimination, partial pivoting is never required—you are asked to provide the details in Exercise 4.3.15.

Problem: In 1900, Minkowski (p. 278) discovered that all diagonally dominant matrices are nonsingular. Establish the validity of Minkowski's result.

Solution: The strategy is to prove that if $\mathbf{A}$ is diagonally dominant, then $N(\mathbf{A}) = \{\mathbf{0}\}$, so that (4.3.2) together with (4.3.6) will provide the desired conclusion. Use an indirect argument—suppose there exists a vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = \mathbf{0}$, and assume that x_k is the entry of maximum magnitude in $\mathbf{x}$. Focus on the k^{th} component of $\mathbf{Ax}$, and write the equation $\mathbf{A}_{k*}\mathbf{x} = 0$ as

$$a_{kk}x_k = -\sum_{\substack{j=1\\j\neq k}}^{n} a_{kj}x_j.$$

Taking absolute values of both sides and using the triangle inequality together with the fact that $|x_j| \leq |x_k|$ for each j produces

$$|a_{kk}|\,|x_k| = \left|\sum_{\substack{j=1\\j\neq k}}^{n} a_{kj}x_j\right| \leq \sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}x_j| = \sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}|\,|x_j| \leq \left(\sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}|\right)|x_k|.$$

But this implies that

$$|a_{kk}| \leq \sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}|,$$

which violates the hypothesis that $\mathbf{A}$ is diagonally dominant. Therefore, the assumption that there exists a nonzero vector in $N(\mathbf{A})$ must be false, so we may conclude that $N(\mathbf{A}) = \{\mathbf{0}\}$, and hence $\mathbf{A}$ is nonsingular.

Note: An alternate solution is given in Example 7.1.6 on p. 499.

Example 4.3.4

Vandermonde Matrices. Matrices of the form

$$\mathbf{V}_{m\times n} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{pmatrix}$$

in which $x_i \neq x_j$ for all $i \neq j$ are called ***Vandermonde***[26] ***matrices.***

Problem: Explain why the columns in $\mathbf{V}$ constitute a linearly independent set whenever $n \leq m$.

Solution: According to (4.3.2), the columns of $\mathbf{V}$ form a linearly independent set if and only if $N(\mathbf{V}) = \{\mathbf{0}\}$. If

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.3.8}$$

then for each $i = 1, 2, \ldots, m$,

$$\alpha_0 + x_i\alpha_1 + x_i^2\alpha_2 + \cdots + x_i^{n-1}\alpha_{n-1} = 0.$$

This implies that the polynomial

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{n-1}x^{n-1}$$

has m distinct roots—namely, the x_i's. However, $\deg p(x) \leq n-1$ and the fundamental theorem of algebra guarantees that if $p(x)$ is not the zero polynomial, then $p(x)$ can have at most $n-1$ distinct roots. Therefore, (4.3.8) holds if and only if $\alpha_i = 0$ for all i, and thus (4.3.2) insures that the columns of $\mathbf{V}$ form a linearly independent set.

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[26] This is named in honor of the French mathematician Alexandre-Theophile Vandermonde (1735–1796). He made a variety of contributions to mathematics, but he is best known perhaps for being the first European to give a logically complete exposition of the theory of determinants. He is regarded by many as being the founder of that theory. However, the matrix $\mathbf{V}$ (and an associated determinant) named after him, by Lebesgue, does not appear in Vandermonde's published work. Vandermonde's first love was music, and he took up mathematics only after he was 35 years old. He advocated the theory that all art and music rested upon a general principle that could be expressed mathematically, and he claimed that almost anyone could become a composer with the aid of mathematics.

Example 4.3.5

Problem: Given a set of m points $\mathcal{S} = \{(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_m, y_m)\}$ in which the x_i 's are distinct, explain why there is a unique polynomial

$$\ell(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_{m-1} t^{m-1} \tag{4.3.9}$$

of degree $m-1$ that passes through each point in $\mathcal{S}$.

Solution: The coefficients α_i must satisfy the equations

$$\begin{aligned}
\alpha_0 + \alpha_1 x_1 + \alpha_2 x_1^2 + \cdots + \alpha_{m-1} x_1^{m-1} &= \ell(x_1) = y_1,\\
\alpha_0 + \alpha_1 x_2 + \alpha_2 x_2^2 + \cdots + \alpha_{m-1} x_2^{m-1} &= \ell(x_2) = y_2,\\
&\ \vdots\\
\alpha_0 + \alpha_1 x_m + \alpha_2 x_m^2 + \cdots + \alpha_{m-1} x_m^{m-1} &= \ell(x_m) = y_m.
\end{aligned}$$

Writing this $m \times m$ system as

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{m-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{m-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{m-1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{m-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

reveals that the coefficient matrix is a square Vandermonde matrix, so the result of Example 4.3.4 guarantees that it is nonsingular. Consequently, the system has a unique solution, and thus there is one and only one possible set of coefficients for the polynomial $\ell(t)$ in (4.3.9). In fact, $\ell(t)$ must be given by

$$\ell(t) = \sum_{i=1}^{m} \left(y_i \frac{\prod_{j \neq i}^{m} (t - x_j)}{\prod_{j \neq i}^{m} (x_i - x_j)} \right).$$

Verify this by showing that the right-hand side is indeed a polynomial of degree $m-1$ that passes through the points in $\mathcal{S}$. The polynomial $\ell(t)$ is known as the ***Lagrange***[27] ***interpolation polynomial*** of degree $m-1$.

If $rank\,(\mathbf{A}_{m \times n}) < n$, then the columns of $\mathbf{A}$ must be a dependent set—recall (4.3.3). For such matrices we often wish to extract a ***maximal linearly independent subset*** of columns—i.e., a linearly independent set containing as many columns from $\mathbf{A}$ as possible. Although there can be several ways to make such a selection, the basic columns in $\mathbf{A}$ always constitute one solution.

[27] Joseph Louis Lagrange (1736–1813), born in Turin, Italy, is considered by many to be one of the two greatest mathematicians of the eighteenth century—Euler is the other. Lagrange occupied Euler's vacated position in 1766 in Berlin at the court of Frederick the Great who wrote that "the greatest king in Europe" wishes to have at his court "the greatest mathematician of Europe." After 20 years, Lagrange left Berlin and eventually moved to France. Lagrange's mathematical contributions are extremely wide and deep, but he had a particularly strong influence on the way mathematical research evolved. He was the first of the top-class mathematicians to recognize the weaknesses in the foundations of calculus, and he was among the first to attempt a rigorous development.

Maximal Independent Subsets

If $rank(\mathbf{A}_{m\times n}) = r$, then the following statements hold.

- Any maximal independent subset of columns from $\mathbf{A}$ contains exactly r columns. (4.3.10)
- Any maximal independent subset of rows from $\mathbf{A}$ contains exactly r rows. (4.3.11)
- In particular, the r basic columns in $\mathbf{A}$ constitute one maximal independent subset of columns from $\mathbf{A}$. (4.3.12)

Proof. Exactly the same linear relationships that exist among the columns of $\mathbf{A}$ must also hold among the columns of $\mathbf{E_A}$—by (3.9.6). This guarantees that a subset of columns from $\mathbf{A}$ is linearly independent if and only if the columns in the corresponding positions in $\mathbf{E_A}$ are an independent set. Let

$$\mathbf{C} = \big(\mathbf{c}_1 \,|\, \mathbf{c}_2 \,|\, \cdots \,|\, \mathbf{c}_k\big)$$

be a matrix that contains an independent subset of columns from $\mathbf{E_A}$ so that $rank(\mathbf{C}) = k$—recall (4.3.3). Since each column in $\mathbf{E_A}$ is a combination of the r basic (unit) columns in $\mathbf{E_A}$, there are scalars β_{ij} such that $\mathbf{c}_j = \sum_{i=1}^{r} \beta_{ij}\mathbf{e}_i$ for $j = 1, 2, \ldots, k$. These equations can be written as the single matrix equation

$$\big(\mathbf{c}_1 \,|\, \mathbf{c}_2 \,|\, \cdots \,|\, \mathbf{c}_k\big) = \big(\mathbf{e}_1 \,|\, \mathbf{e}_2 \,|\, \cdots \,|\, \mathbf{e}_r\big) \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1k} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{r1} & \beta_{r2} & \cdots & \beta_{rk} \end{pmatrix}$$

or

$$\mathbf{C}_{m\times k} = \begin{pmatrix} \mathbf{I}_r \\ \mathbf{0} \end{pmatrix} \mathbf{B}_{r\times k} = \begin{pmatrix} \mathbf{B}_{r\times k} \\ \mathbf{0} \end{pmatrix}, \quad \text{where} \quad \mathbf{B} = [\beta_{ij}].$$

Consequently, $r \geq rank(\mathbf{C}) = k$, and therefore any independent subset of columns from $\mathbf{E_A}$—and hence any independent set of columns from $\mathbf{A}$—cannot contain more than r vectors. Because the r basic (unit) columns in $\mathbf{E_A}$ form an independent set, the r basic columns in $\mathbf{A}$ constitute an independent set. This proves (4.3.10) and (4.3.12). The proof of (4.3.11) follows from the fact that $rank(\mathbf{A}) = rank\left(\mathbf{A}^T\right)$—recall (3.9.11). ∎

Basic Facts of Independence

For a nonempty set of vectors $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ in a space $\mathcal{V}$, the following statements are true.

- If $\mathcal{S}$ contains a linearly dependent subset, then $\mathcal{S}$ itself must be linearly dependent. (4.3.13)
- If $\mathcal{S}$ is linearly independent, then every subset of $\mathcal{S}$ is also linearly independent. (4.3.14)
- If $\mathcal{S}$ is linearly independent and if $\mathbf{v} \in \mathcal{V}$, then the ***extension set*** $\mathcal{S}_{ext} = \mathcal{S} \cup \{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \notin span(\mathcal{S})$. (4.3.15)
- If $\mathcal{S} \subseteq \Re^m$ and if $n > m$, then $\mathcal{S}$ must be linearly dependent. (4.3.16)

Proof of (4.3.13). Suppose that $\mathcal{S}$ contains a linearly dependent subset, and, for the sake of convenience, suppose that the vectors in $\mathcal{S}$ have been permuted so that this dependent subset is $\mathcal{S}_{dep} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$. According to the definition of dependence, there must be scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$, not all of which are zero, such that $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k = \mathbf{0}$. This means that we can write

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k + 0\mathbf{u}_{k+1} + \cdots + 0\mathbf{u}_n = \mathbf{0},$$

where not all of the scalars are zero, and hence $\mathcal{S}$ is linearly dependent.

Proof of (4.3.14). This is an immediate consequence of (4.3.13).

Proof of (4.3.15). If $\mathcal{S}_{ext}$ is linearly independent, then $\mathbf{v} \notin span(\mathcal{S})$, for otherwise $\mathbf{v}$ would be a combination of vectors from $\mathcal{S}$ thus forcing $\mathcal{S}_{ext}$ to be a dependent set. Conversely, suppose $\mathbf{v} \notin span(\mathcal{S})$. To prove that $\mathcal{S}_{ext}$ is linearly independent, consider a linear combination

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n + \alpha_{n+1}\mathbf{v} = \mathbf{0}. \qquad (4.3.17)$$

It must be the case that $\alpha_{n+1} = 0$, for otherwise $\mathbf{v}$ would be a combination of vectors from $\mathcal{S}$. Consequently,

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}.$$

But this implies that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

because $\mathcal{S}$ is linearly independent. Therefore, the only solution for the α's in (4.3.17) is the trivial set, and hence $\mathcal{S}_{ext}$ must be linearly independent.

Proof of (4.3.16). This follows from (4.3.3) because if the $\mathbf{u}_i$'s are placed as columns in a matrix $\mathbf{A}_{m\times n}$, then $rank(\mathbf{A}) \leq m < n$. ∎

Example 4.3.6

Let $\mathcal{V}$ be the vector space of real-valued functions of a real variable, and let $\mathcal{S} = \{f_1(x), f_2(x), \ldots, f_n(x)\}$ be a set of functions that are $n-1$ times differentiable. The ***Wronski***[28] ***matrix*** is defined to be

$$\mathbf{W}(x) = \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}.$$

Problem: If there is at least one point $x = x_0$ such that $\mathbf{W}(x_0)$ is nonsingular, prove that $\mathcal{S}$ must be a linearly independent set.

Solution: Suppose that

$$0 = \alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_n f_n(x) \tag{4.3.18}$$

for all values of x. When $x = x_0$, it follows that

$$\begin{aligned} 0 &= \alpha_1 f_1(x_0) + \alpha_2 f_2(x_0) + \cdots + \alpha_n f_n(x_0), \\ 0 &= \alpha_1 f_1'(x_0) + \alpha_2 f_2'(x_0) + \cdots + \alpha_n f_n'(x_0), \\ &\vdots \\ 0 &= \alpha_1 f_1^{(n-1)}(x_0) + \alpha_2 f_2^{(n-1)}(x_0) + \cdots + \alpha_n f_n^{(n-1)}(x_0), \end{aligned}$$

which means that $\mathbf{v} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in N\big(\mathbf{W}(x_0)\big)$. But $N\big(\mathbf{W}(x_0)\big) = \{\mathbf{0}\}$ because $\mathbf{W}(x_0)$ is nonsingular, and hence $\mathbf{v} = \mathbf{0}$. Therefore, the only solution for the α's in (4.3.18) is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ thereby insuring that $\mathcal{S}$ is linearly independent.

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[28] This matrix is named in honor of the Polish mathematician Jozef Maria Höené Wronski (1778–1853), who studied four special forms of determinants, one of which was the determinant of the matrix that bears his name. Wronski was born to a poor family near Poznan, Poland, but he studied in Germany and spent most of his life in France. He is reported to have been an egotistical person who wrote in an exhaustively wearisome style. Consequently, almost no one read his work. Had it not been for his lone follower, Ferdinand Schweins (1780–1856) of Heidelberg, Wronski would probably be unknown today. Schweins preserved and extended Wronski's results in his own writings, which in turn received attention from others. Wronski also wrote on philosophy. While trying to reconcile Kant's metaphysics with Leibniz's calculus, Wronski developed a social philosophy called "Messianism" that was based on the belief that absolute truth could be achieved through mathematics.

For example, to verify that the set of polynomials $\mathcal{P} = \{1, x, x^2, \ldots, x^n\}$ is linearly independent, observe that the associated Wronski matrix

$$\mathbf{W}(x) = \begin{pmatrix} 1 & x & x^2 & \cdots & x^n \\ 0 & 1 & 2x & \cdots & nx^{n-1} \\ 0 & 0 & 2 & \cdots & n(n-1)x^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n! \end{pmatrix}$$

is triangular with nonzero diagonal entries. Consequently, $\mathbf{W}(x)$ is nonsingular for every value of x, and hence $\mathcal{P}$ must be an independent set.

Exercises for section 4.3

4.3.1. Determine which of the following sets are linearly independent. For those sets that are linearly dependent, write one of the vectors as a linear combination of the others.

(a) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 9 \end{pmatrix} \right\}$,

(b) $\{(1 \;\; 2 \;\; 3), (0 \;\; 4 \;\; 5), (0 \;\; 0 \;\; 6), (1 \;\; 1 \;\; 1)\}$,

(c) $\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$,

(d) $\{(2 \;\; 2 \;\; 2 \;\; 2), (2 \;\; 2 \;\; 0 \;\; 2), (2 \;\; 0 \;\; 2 \;\; 2)\}$,

(e) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 4 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\}$.

4.3.2. Consider the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 2 & 1 & 2 \\ 6 & 3 & 2 & 2 \end{pmatrix}$.

(a) Determine a maximal linearly independent subset of columns from $\mathbf{A}$.

(b) Determine the total number of linearly independent subsets that can be constructed using the columns of $\mathbf{A}$.

4.3.3. Suppose that in a population of a million children the height of each one is measured at ages 1 year, 2 years, and 3 years, and accumulate this data in a matrix

$$\begin{array}{c} \\ \#1 \\ \#2 \\ \vdots \\ \#i \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} 1\text{ yr} & 2\text{ yr} & 3\text{ yr} \end{array} \\ \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ \vdots & \vdots & \vdots \\ h_{i1} & h_{i2} & h_{i3} \\ \vdots & \vdots & \vdots \end{pmatrix} \end{array} = \mathbf{H}.$$

Explain why there are at most three "independent children" in the sense that the heights of all the other children must be a combination of these "independent" ones.

4.3.4. Consider a particular species of wildflower in which each plant has several stems, leaves, and flowers, and for each plant let the following hold.

$S =$ the average stem length (in inches).
$L =$ the average leaf width (in inches).
$F =$ the number of flowers.

Four particular plants are examined, and the information is tabulated in the following matrix:

$$\mathbf{A} = \begin{array}{c} \\ \#1 \\ \#2 \\ \#3 \\ \#4 \end{array} \begin{array}{c} \begin{array}{ccc} S & L & F \end{array} \\ \begin{pmatrix} 1 & 1 & 10 \\ 2 & 1 & 12 \\ 2 & 2 & 15 \\ 3 & 2 & 17 \end{pmatrix} \end{array}.$$

For these four plants, determine whether or not there exists a linear relationship between S, L, and F. In other words, do there exist constants $\alpha_0, \alpha_1, \alpha_2$, and α_3 such that $\alpha_0 + \alpha_1 S + \alpha_2 L + \alpha_3 F = 0$?

4.3.5. Let $\mathcal{S} = \{\mathbf{0}\}$ be the set containing only the zero vector.

(a) Explain why $\mathcal{S}$ must be linearly dependent.
(b) Explain why any set containing a zero vector must be linearly dependent.

4.3.6. If $\mathbf{T}$ is a triangular matrix in which each $t_{ii} \neq 0$, explain why the rows and columns of $\mathbf{T}$ must each be linearly independent sets.

4.3.7. Determine whether or not the following set of matrices is a linearly independent set:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

4.3.8. Without doing any computation, determine whether the following matrix is singular or nonsingular:

$$\mathbf{A} = \begin{pmatrix} n & 1 & 1 & \cdots & 1 \\ 1 & n & 1 & \cdots & 1 \\ 1 & 1 & n & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & n \end{pmatrix}_{n \times n}.$$

4.3.9. In theory, determining whether or not a given set is linearly independent is a well-defined problem with a straightforward solution. In practice, however, this problem is often not so well defined because it becomes clouded by the fact that we usually cannot use exact arithmetic, and contradictory conclusions may be produced depending upon the precision of the arithmetic. For example, let

$$\mathcal{S} = \left\{ \begin{pmatrix} .1 \\ .4 \\ .7 \end{pmatrix}, \begin{pmatrix} .2 \\ .5 \\ .8 \end{pmatrix}, \begin{pmatrix} .3 \\ .6 \\ .901 \end{pmatrix} \right\}.$$

(a) Use exact arithmetic to determine whether or not $\mathcal{S}$ is linearly independent.

(b) Use 3-digit arithmetic (without pivoting or scaling) to determine whether or not $\mathcal{S}$ is linearly independent.

4.3.10. If $\mathbf{A}_{m\times n}$ is a matrix such that $\sum_{j=1}^{n} a_{ij} = 0$ for each $i = 1, 2, \ldots, m$ (i.e., each row sum is 0), explain why the columns of $\mathbf{A}$ are a linearly dependent set, and hence $rank(\mathbf{A}) < n$.

4.3.11. If $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a linearly independent subset of $\Re^{m\times 1}$, and if $\mathbf{P}_{m\times m}$ is a nonsingular matrix, explain why the set

$$\mathbf{P}(\mathcal{S}) = \{\mathbf{Pu}_1, \mathbf{Pu}_2, \ldots, \mathbf{Pu}_n\}$$

must also be a linearly independent set. Is this result still true if $\mathbf{P}$ is singular?

4.3.12. Suppose that $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a set of vectors from $\Re^m$. Prove that $\mathcal{S}$ is linearly independent if and only if the set

$$\mathcal{S}' = \left\{ \mathbf{u}_1, \ \sum_{i=1}^{2} \mathbf{u}_i, \ \sum_{i=1}^{3} \mathbf{u}_i, \ \ldots, \ \sum_{i=1}^{n} \mathbf{u}_i \right\}$$

is linearly independent.

4.3.13. Which of the following sets of functions are linearly independent?

(a) $\{\sin x, \ \cos x, \ x \sin x\}$.

(b) $\left\{e^x, \ xe^x, \ x^2 e^x\right\}$.

(c) $\left\{\sin^2 x, \ \cos^2 x, \ \cos 2x\right\}$.

4.3.14. Prove that the converse of the statement given in Example 4.3.6 is false by showing that $\mathcal{S} = \left\{x^3, \ |x|^3\right\}$ is a linearly independent set, but the associated Wronski matrix $\mathbf{W}(x)$ is singular for all values of x.

4.3.15. If $\mathbf{A}^T$ is diagonally dominant, explain why partial pivoting is not needed when solving $\mathbf{Ax} = \mathbf{b}$ by Gaussian elimination. **Hint:** If after one step of Gaussian elimination we have

$$\mathbf{A} = \begin{pmatrix} \alpha & \mathbf{d}^T \\ \mathbf{c} & \mathbf{B} \end{pmatrix} \xrightarrow{\textit{one step}} \begin{pmatrix} \alpha & \mathbf{d}^T \\ \mathbf{0} & \mathbf{B} - \frac{\mathbf{c}\mathbf{d}^T}{\alpha} \end{pmatrix},$$

show that $\mathbf{A}^T$ being diagonally dominant implies $\mathbf{X} = \left(\mathbf{B} - \frac{\mathbf{c}\mathbf{d}^T}{\alpha}\right)^T$ must also be diagonally dominant.

4.2.8. $\mathbf{y}^T\mathbf{b} = 0 \ \forall\ \mathbf{y} \in N\left(\mathbf{A}^T\right) = R\left(\mathbf{P}_2^T\right) \implies \mathbf{P}_2\mathbf{b} = 0 \implies \mathbf{b} \in N(\mathbf{P}_2) = R(\mathbf{A})$

4.2.9. $\mathbf{x} \in R\big(\mathbf{A} \mid \mathbf{B}\big) \iff \exists\ \mathbf{y}$ such that $\mathbf{x} = \big(\mathbf{A} \mid \mathbf{B}\big)\mathbf{y} = \big(\mathbf{A} \mid \mathbf{B}\big)\begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{pmatrix} = \mathbf{A}\mathbf{y}_1 + \mathbf{B}\mathbf{y}_2 \iff \mathbf{x} \in R(\mathbf{A}) + R(\mathbf{B})$

4.2.10. (a) $\mathbf{p} + N(\mathbf{A})$ is the set of all possible solutions to $\mathbf{A}\mathbf{x} = \mathbf{b}$. Recall from (2.5.7) that the general solution of a nonhomogeneous equation is a particular solution plus the general solution of the homogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{0}$. The general solution of the homogeneous equation is simply a way of describing all possible solutions of $\mathbf{A}\mathbf{x} = \mathbf{0}$, which is $N(\mathbf{A})$.

(b) $rank(\mathbf{A}_{3\times 3}) = 1$ means that $N(\mathbf{A})$ is spanned by two vectors, and hence $N(\mathbf{A})$ is a plane through the origin. From the parallelogram law, $\mathbf{p} + N(\mathbf{A})$ is a plane parallel to $N(\mathbf{A})$ passing through the point defined by $\mathbf{p}$.

(c) This time $N(\mathbf{A})$ is spanned by a single vector, and $\mathbf{p} + N(\mathbf{A})$ is a line parallel to $N(\mathbf{A})$ passing through the point defined by $\mathbf{p}$.

4.2.11. $\mathbf{a} \in R\left(\mathbf{A}^T\right) \iff \exists\ \mathbf{y}$ such that $\mathbf{a}^T = \mathbf{y}^T\mathbf{A}$. If $\mathbf{A}\mathbf{x} = \mathbf{b}$, then

$$\mathbf{a}^T\mathbf{x} = \mathbf{y}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{b},$$

which is independent of $\mathbf{x}$.

4.2.12. (a) $\mathbf{b} \in R(\mathbf{AB}) \implies \exists\ \mathbf{x}$ such that $\mathbf{b} = \mathbf{ABx} = \mathbf{A}(\mathbf{Bx}) \implies \mathbf{b} \in R(\mathbf{A})$ because $\mathbf{b}$ is the image of $\mathbf{Bx}$.

(b) $\mathbf{x} \in N(\mathbf{B}) \implies \mathbf{Bx} = \mathbf{0} \implies \mathbf{ABx} = \mathbf{0} \implies \mathbf{x} \in N(\mathbf{AB})$.

4.2.13. Given any $\mathbf{z} \in R(\mathbf{AB})$, the object is to show that $\mathbf{z}$ can be written as some linear combination of the $\mathbf{A}\mathbf{b}_i$'s. Argue as follows. $\mathbf{z} \in R(\mathbf{AB}) \implies \mathbf{z} = \mathbf{ABy}$ for some $\mathbf{y}$. But it is always true that $\mathbf{By} \in R(\mathbf{B})$, so

$$\mathbf{By} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \cdots + \alpha_n\mathbf{b}_n,$$

and therefore $\mathbf{z} = \mathbf{ABy} = \alpha_1\mathbf{A}\mathbf{b}_1 + \alpha_2\mathbf{A}\mathbf{b}_2 + \cdots + \alpha_n\mathbf{A}\mathbf{b}_n$.

Solutions for exercises in section 4. 3

4.3.1. (a) and (b) are linearly dependent—all others are linearly independent. To write one vector as a combination of others in a dependent set, place the vectors as columns in $\mathbf{A}$ and find $\mathbf{E}_{\mathbf{A}}$. This reveals the dependence relationships among columns of $\mathbf{A}$.

4.3.2. (a) According to (4.3.12), the basic columns in $\mathbf{A}$ always constitute one maximal linearly independent subset.

(b) Ten—5 sets using two vectors, 4 sets using one vector, and the empty set.

4.3.3. $rank(\mathbf{H}) \leq 3$, and according to (4.3.11), $rank(\mathbf{H})$ is the maximal number of independent rows in $\mathbf{H}$.

4.3.4. The question is really whether or not the columns in

$$\hat{\mathbf{A}} = \begin{array}{c} \\ \#1 \\ \#2 \\ \#3 \\ \#4 \end{array} \begin{array}{c} \begin{array}{cccc} & S & L & F \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 10 \\ 1 & 2 & 1 & 12 \\ 1 & 2 & 2 & 15 \\ 1 & 3 & 2 & 17 \end{pmatrix} \end{array}$$

are linearly independent. Reducing $\hat{\mathbf{A}}$ to $\mathbf{E}_{\hat{\mathbf{A}}}$ shows that $5+2S+3L-F=0$.

4.3.5. (a) This follows directly from the definition of linear dependence because there are nonzero values of α such that $\alpha\mathbf{0}=\mathbf{0}$.

(b) This is a consequence of (4.3.13).

4.3.6. If each $t_{ii} \neq 0$, then $\mathbf{T}$ is nonsingular, and the result follows from (4.3.6) and (4.3.7).

4.3.7. It is linearly independent because

$$\alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Longleftrightarrow \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

4.3.8. $\mathbf{A}$ is nonsingular because it is diagonally dominant.

4.3.9. $\mathcal{S}$ is linearly independent using exact arithmetic, but using 3-digit arithmetic yields the conclusion that $\mathcal{S}$ is dependent.

4.3.10. If $\mathbf{e}$ is the column vector of all 1's, then $\mathbf{Ae}=\mathbf{0}$, so that $N(\mathbf{A}) \neq \{\mathbf{0}\}$.

4.3.11. (Solution 1.) $\sum_i \alpha_i \mathbf{Pu}_i = \mathbf{0} \implies \mathbf{P}\sum_i \alpha_i \mathbf{u}_i = \mathbf{0} \implies \sum_i \alpha_i \mathbf{u}_i = \mathbf{0} \implies$ each $\alpha_i = 0$ because the $\mathbf{u}_i$'s are linearly independent.

(Solution 2.) If $\mathbf{A}_{m\times n}$ is the matrix containing the $\mathbf{u}_i$'s as columns, then $\mathbf{PA}=\mathbf{B}$ is the matrix containing the vectors in $\mathbf{P}(\mathcal{S})$ as its columns. Now,

$$\mathbf{A} \overset{\text{row}}{\sim} \mathbf{B} \implies rank\,(\mathbf{B}) = rank\,(\mathbf{A}) = n,$$

and hence (4.3.3) insures that the columns of $\mathbf{B}$ are linearly independent. The result need not be true if $\mathbf{P}$ is singular—take $\mathbf{P}=\mathbf{0}$ for example.

4.3.12. If $\mathbf{A}_{m\times n}$ is the matrix containing the $\mathbf{u}_i$'s as columns, and if

$$\mathbf{Q}_{n\times n} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

then the columns of $\mathbf{B} = \mathbf{AQ}$ are the vectors in $\mathcal{S}'$. Clearly, $\mathbf{Q}$ is nonsingular so that $\mathbf{A} \overset{\text{col}}{\sim} \mathbf{B}$, and thus $rank\,(\mathbf{A}) = rank\,(\mathbf{B})$. The desired result now follows from (4.3.3).

4.3.13. (a) and (b) are linearly independent because the Wronski matrix $\mathbf{W}(0)$ is nonsingular in each case. (c) is dependent because $\sin^2 x - \cos^2 x + \cos 2x = 0$.

4.3.14. If $\mathcal{S}$ were dependent, then there would exist a constant α such that $x^3 = \alpha|x|^3$ for all values of x. But this would mean that

$$\alpha = \frac{x^3}{|x|^3} = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases}$$

which is clearly impossible since α must be constant. The associated Wronski matrix is

$$\mathbf{W}(x) = \begin{cases} \begin{pmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{pmatrix} & \text{when } x \geq 0, \\ \begin{pmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{pmatrix} & \text{when } x < 0, \end{cases}$$

which is singular for all values of x.

4.3.15. Start with the fact that

$$\begin{aligned} \mathbf{A}^T \text{ diag. dom.} \quad &\Longrightarrow \quad |b_{ii}| > |d_i| + \sum_{j\neq i} |b_{ji}| \quad \text{and} \quad |\alpha| > \sum_j |c_j| \\ &\Longrightarrow \quad \sum_{j\neq i} |b_{ji}| < |b_{ii}| - |d_i| \quad \text{and} \quad \frac{1}{|\alpha|}\sum_{j\neq i} |c_j| < 1 - \frac{|c_i|}{|\alpha|}, \end{aligned}$$

and then use the forward and backward triangle inequality to write

$$\begin{aligned} \sum_{j\neq i} |x_{ij}| &= \sum_{j\neq i} \left| b_{ji} - \frac{d_i c_j}{\alpha} \right| \leq \sum_{j\neq i} |b_{ji}| + \frac{|d_i|}{|\alpha|} \sum_{j\neq i} |c_j| \\ &< \big(|b_{ii}| - |d_i|\big) + |d_i| \left(1 - \frac{|c_i|}{|\alpha|} \right) = |b_{ii}| - \frac{|d_i|\,|c_i|}{|\alpha|} \\ &\leq \left| b_{ii} - \frac{d_i c_i}{\alpha} \right| = |x_{ii}|. \end{aligned}$$

Now, diagonal dominance of $\mathbf{A}^T$ insures that α is the entry of largest magnitude in the first column of $\mathbf{A}$, so no row interchange is needed at the first step of Gaussian elimination. After one step, the diagonal dominance of $\mathbf{X}$ guarantees that the magnitude of the second pivot is maximal with respect to row interchanges. Proceeding by induction establishes that no step requires partial pivoting.