4.4 BASIS AND DIMENSION

Recall from §4.1 that $\mathcal{S}$ is a spanning set for a space $\mathcal{V}$ if and only if every vector in $\mathcal{V}$ is a linear combination of vectors in $\mathcal{S}$. However, spanning sets can contain redundant vectors. For example, a subspace $\mathcal{L}$ defined by a line through the origin in $\Re^2$ may be spanned by any number of nonzero vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathcal{L}$, but any one of the vectors $\{\mathbf{v}_i\}$ by itself will suffice. Similarly, a plane $\mathcal{P}$ through the origin in $\Re^3$ can be spanned in many different ways, but the parallelogram law indicates that a minimal spanning set need only be an independent set of two vectors from $\mathcal{P}$. These considerations motivate the following definition.

Basis

A linearly independent spanning set for a vector space $\mathcal{V}$ is called a ***basis*** for $\mathcal{V}$.

It can be proven that every vector space $\mathcal{V}$ possesses a basis—details for the case when $\mathcal{V} \subseteq \Re^m$ are asked for in the exercises. Just as in the case of spanning sets, a space can possess many different bases.

Example 4.4.1

- The unit vectors $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in $\Re^n$ are a basis for $\Re^n$. This is called the ***standard basis*** for $\Re^n$.

- If $\mathbf{A}$ is an $n \times n$ nonsingular matrix, then the set of rows in $\mathbf{A}$ as well as the set of columns from $\mathbf{A}$ constitute a basis for $\Re^n$. For example, (4.3.3) insures that the columns of $\mathbf{A}$ are linearly independent, and we know they span $\Re^n$ because $R(\mathbf{A}) = \Re^n$ —recall Exercise 4.2.5(b).

- For the trivial vector space $\mathcal{Z} = \{\mathbf{0}\}$, there is no nonempty linearly independent spanning set. Consequently, the empty set is considered to be a basis for $\mathcal{Z}$.

- The set $\{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space of polynomials having degree n or less.

- The infinite set $\{1, x, x^2, \ldots\}$ is a basis for the vector space of all polynomials. It should be clear that no finite basis is possible.

Spaces that possess a basis containing an infinite number of vectors are referred to as ***infinite-dimensional spaces,*** and those that have a finite basis are called ***finite-dimensional spaces.*** This is often a line of demarcation in the study of vector spaces. A complete theoretical treatment would include the analysis of infinite-dimensional spaces, but this text is primarily concerned with finite-dimensional spaces over the real or complex numbers. It can be shown that, in effect, this amounts to analyzing $\Re^n$ or $\mathcal{C}^n$ and their subspaces.

The original concern of this section was to try to eliminate redundancies from spanning sets so as to provide spanning sets containing a minimal number of vectors. The following theorem shows that a basis is indeed such a set.

Characterizations of a Basis

Let $\mathcal{V}$ be a subspace of $\Re^m$, and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \subseteq \mathcal{V}$. The following statements are equivalent.

- $\mathcal{B}$ is a basis for $\mathcal{V}$. (4.4.1)
- $\mathcal{B}$ is a minimal spanning set for $\mathcal{V}$. (4.4.2)
- $\mathcal{B}$ is a maximal linearly independent subset of $\mathcal{V}$. (4.4.3)

Proof. First argue that (4.4.1) $\Longrightarrow$ (4.4.2) $\Longrightarrow$ (4.4.1), and then show (4.4.1) is equivalent to (4.4.3).

Proof of (4.4.1) $\Longrightarrow$ (4.4.2). First suppose that $\mathcal{B}$ is a basis for $\mathcal{V}$, and prove that $\mathcal{B}$ is a minimal spanning set by using an indirect argument—i.e., assume that $\mathcal{B}$ is *not* minimal, and show that this leads to a contradiction. If $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is a basis for $\mathcal{V}$ in which $k < n$, then each $\mathbf{b}_j$ can be written as a combination of the $\mathbf{x}_i$'s. That is, there are scalars α_{ij} such that

$$\mathbf{b}_j = \sum_{i=1}^{k} \alpha_{ij}\mathbf{x}_i \quad \text{for } j = 1, 2, \ldots, n. \tag{4.4.4}$$

If the $\mathbf{b}$'s and $\mathbf{x}$'s are placed as columns in matrices

$$\mathbf{B}_{m\times n} = \big(\mathbf{b}_1 \,|\, \mathbf{b}_2 \,|\, \cdots \,|\, \mathbf{b}_n\big) \quad \text{and} \quad \mathbf{X}_{m\times k} = \big(\mathbf{x}_1 \,|\, \mathbf{x}_2 \,|\, \cdots \,|\, \mathbf{x}_k\big),$$

then (4.4.4) can be expressed as the matrix equation

$$\mathbf{B} = \mathbf{XA}, \quad \text{where,} \quad \mathbf{A}_{k\times n} = [\alpha_{ij}]\,.$$

Since the rank of a matrix cannot exceed either of its size dimensions, and since $k < n$, we have that $rank\,(\mathbf{A}) \le k < n$, so that $N\,(\mathbf{A}) \ne \{\mathbf{0}\}$—recall (4.2.10). If $\mathbf{z} \ne \mathbf{0}$ is such that $\mathbf{Az} = \mathbf{0}$, then $\mathbf{Bz} = \mathbf{0}$. But this is impossible because the columns of $\mathbf{B}$ are linearly independent, and hence $N(\mathbf{B}) = \{\mathbf{0}\}$—recall (4.3.2). Therefore, the supposition that there exists a basis for $\mathcal{V}$ containing fewer than n vectors must be false, and we may conclude that $\mathcal{B}$ is indeed a minimal spanning set.

Proof of (4.4.2) $\Longrightarrow$ (4.4.1). If $\mathcal{B}$ is a minimal spanning set, then $\mathcal{B}$ must be a *linearly independent* spanning set. Otherwise, some $\mathbf{b}_i$ would be a linear combination of the other $\mathbf{b}$'s, and the set

$$\mathcal{B}' = \{\mathbf{b}_1, \ldots, \mathbf{b}_{i-1}, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_n\}$$

would still span $\mathcal{V}$, but $\mathcal{B}'$ would contain fewer vectors than $\mathcal{B}$, which is impossible because $\mathcal{B}$ is a *minimal* spanning set.

Proof of (4.4.3) $\Longrightarrow$ (4.4.1). If $\mathcal{B}$ is a maximal linearly independent subset of $\mathcal{V}$, but not a basis for $\mathcal{V}$, then there exists a vector $\mathbf{v} \in \mathcal{V}$ such that $\mathbf{v} \notin span(\mathcal{B})$. This means that the extension set

$$\mathcal{B} \cup \{\mathbf{v}\} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n, \mathbf{v}\}$$

is linearly independent—recall (4.3.15). But this is impossible because $\mathcal{B}$ is a *maximal* linearly independent subset of $\mathcal{V}$. Therefore, $\mathcal{B}$ is a basis for $\mathcal{V}$.

Proof of (4.4.1) $\Longrightarrow$ (4.4.3). Suppose that $\mathcal{B}$ is a basis for $\mathcal{V}$, but not a maximal linearly independent subset of $\mathcal{V}$, and let

$$\mathcal{Y} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\} \subseteq \mathcal{V}, \quad \text{where} \quad k > n$$

be a maximal linearly independent subset—recall that (4.3.16) insures the existence of such a set. The previous argument shows that $\mathcal{Y}$ must be a basis for $\mathcal{V}$. But this is impossible because we already know that a basis must be a minimal spanning set, and $\mathcal{B}$ is a spanning set containing fewer vectors than $\mathcal{Y}$. Therefore, $\mathcal{B}$ must be a maximal linearly independent subset of $\mathcal{V}$. ∎

Although a space $\mathcal{V}$ can have many different bases, the preceding result guarantees that all bases for $\mathcal{V}$ contain the same number of vectors. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are each a basis for $\mathcal{V}$, then each is a minimal spanning set, and thus they must contain the same number of vectors. As we are about to see, this number is quite important.

Dimension

The ***dimension*** of a vector space $\mathcal{V}$ is defined to be

$$\begin{aligned} \dim \mathcal{V} &= \text{number of vectors in any basis for } \mathcal{V} \\ &= \text{number of vectors in any minimal spanning set for } \mathcal{V} \\ &= \text{number of vectors in any maximal independent subset of } \mathcal{V}. \end{aligned}$$

Example 4.4.2

- If $\mathcal{Z} = \{\mathbf{0}\}$ is the trivial subspace, then $\dim \mathcal{Z} = 0$ because the basis for this space is the empty set.

- If $\mathcal{L}$ is a line through the origin in $\Re^3$, then $\dim \mathcal{L} = 1$ because a basis for $\mathcal{L}$ consists of any nonzero vector lying along $\mathcal{L}$.
- If $\mathcal{P}$ is a plane through the origin in $\Re^3$, then $\dim \mathcal{P} = 2$ because a minimal spanning set for $\mathcal{P}$ must contain two vectors from $\mathcal{P}$.
- $\dim \Re^3 = 3$ because the three unit vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ constitute a basis for $\Re^3$.
- $\dim \Re^n = n$ because the unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in $\Re^n$ form a basis.

Example 4.4.3

Problem: If $\mathcal{V}$ is an n-dimensional space, explain why every independent subset $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathcal{V}$ containing n vectors must be a basis for $\mathcal{V}$.

Solution: $\dim \mathcal{V} = n$ means that every subset of $\mathcal{V}$ that contains more than n vectors must be linearly dependent. Consequently, $\mathcal{S}$ is a maximal independent subset of $\mathcal{V}$, and hence $\mathcal{S}$ is a basis for $\mathcal{V}$.

Example 4.4.2 shows that in a loose sense the dimension of a space is a measure of the amount of "stuff" in the space—a plane $\mathcal{P}$ in $\Re^3$ has more "stuff" in it than a line $\mathcal{L}$, but $\mathcal{P}$ contains less "stuff" than the entire space $\Re^3$. Recall from the discussion in §4.1 that subspaces of $\Re^n$ are generalized versions of flat surfaces through the origin. The concept of dimension gives us a way to distinguish between these "flat" objects according to how much "stuff" they contain—much the same way we distinguish between lines and planes in $\Re^3$. Another way to think about dimension is in terms of "degrees of freedom." In the trivial space $\mathcal{Z}$, there are no degrees of freedom—you can move nowhere—whereas on a line there is one degree of freedom—length; in a plane there are two degrees of freedom—length and width; in $\Re^3$ there are three degrees of freedom—length, width, and height; etc.

It is important not to confuse the dimension of a vector space $\mathcal{V}$ with the number of components contained in the individual vectors from $\mathcal{V}$. For example, if $\mathcal{P}$ is a plane through the origin in $\Re^3$, then $\dim \mathcal{P} = 2$, but the individual vectors in $\mathcal{P}$ each have three components. Although the dimension of a space $\mathcal{V}$ and the number of components contained in the individual vectors from $\mathcal{V}$ need not be the same, they are nevertheless related. For example, if $\mathcal{V}$ is a subspace of $\Re^n$, then (4.3.16) insures that no linearly independent subset in $\mathcal{V}$ can contain more than n vectors and, consequently, $\dim \mathcal{V} \le n$. This observation generalizes to produce the following theorem.

Subspace Dimension

For vector spaces $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \subseteq \mathcal{N}$, the following statements are true.

- $\dim \mathcal{M} \leq \dim \mathcal{N}$. (4.4.5)
- If $\dim \mathcal{M} = \dim \mathcal{N}$, then $\mathcal{M} = \mathcal{N}$. (4.4.6)

Proof. Let $\dim \mathcal{M} = m$ and $\dim \mathcal{N} = n$, and use an indirect argument to prove (4.4.5). If it were the case that $m > n$, then there would exist a linearly independent subset of $\mathcal{N}$ (namely, a basis for $\mathcal{M}$) containing more than n vectors. But this is impossible because $\dim \mathcal{N}$ is the size of a maximal independent subset of $\mathcal{N}$. Thus $m \leq n$. Now prove (4.4.6). If $m = n$ but $\mathcal{M} \neq \mathcal{N}$, then there exists a vector $\mathbf{x}$ such that $\mathbf{x} \in \mathcal{N}$ but $\mathbf{x} \notin \mathcal{M}$. If $\mathcal{B}$ is a basis for $\mathcal{M}$, then $\mathbf{x} \notin span(\mathcal{B})$, and the extension set $\mathcal{E} = \mathcal{B} \cup \{\mathbf{x}\}$ is a linearly independent subset of $\mathcal{N}$—recall (4.3.15). But $\mathcal{E}$ contains $m + 1 = n + 1$ vectors, which is impossible because $\dim \mathcal{N} = n$ is the size of a maximal independent subset of $\mathcal{N}$. Hence $\mathcal{M} = \mathcal{N}$. ∎

Let's now find bases and dimensions for the four fundamental subspaces of an $m \times n$ matrix $\mathbf{A}$ of rank r, and let's start with $R(\mathbf{A})$. The entire set of columns in $\mathbf{A}$ spans $R(\mathbf{A})$, but they won't form a basis when there are dependencies among some of the columns. However, the set of *basic* columns in $\mathbf{A}$ is also a spanning set—recall (4.2.8)—and the basic columns always constitute a linearly independent set because no basic column can be a combination of other basic columns (otherwise it wouldn't be basic). So, the set of basic columns is a basis for $R(\mathbf{A})$, and, since there are r of them, $\dim R(\mathbf{A}) = r = rank(\mathbf{A})$.

Similarly, the entire set of rows in $\mathbf{A}$ spans $R(\mathbf{A}^T)$, but the set of all rows is not a basis when dependencies exist. Recall from (4.2.7) that if $\mathbf{U} = \begin{pmatrix} \mathbf{C}_{r \times n} \\ \mathbf{0} \end{pmatrix}$ is any row echelon form that is row equivalent to $\mathbf{A}$, then the rows of $\mathbf{C}$ span $R(\mathbf{A}^T)$. Since $rank(\mathbf{C}) = r$, (4.3.5) insures that the rows of $\mathbf{C}$ are linearly independent. Consequently, the rows in $\mathbf{C}$ are a basis for $R(\mathbf{A}^T)$, and, since there are r of them, $\dim R(\mathbf{A}^T) = r = rank(\mathbf{A})$. Older texts referred to $\dim R(\mathbf{A}^T)$ as the *row rank* of $\mathbf{A}$, while $\dim R(\mathbf{A})$ was called the *column rank* of $\mathbf{A}$, and it was a major task to prove that the row rank always agrees with the column rank. Notice that this is a consequence of the discussion above where it was observed that $\dim R(\mathbf{A}^T) = r = \dim R(\mathbf{A})$.

Turning to the nullspaces, let's first examine $N(\mathbf{A}^T)$. We know from (4.2.12) that if $\mathbf{P}$ is a nonsingular matrix such that $\mathbf{PA} = \mathbf{U}$ is in row echelon form, then the last $m - r$ rows in $\mathbf{P}$ span $N(\mathbf{A}^T)$. Because the set of rows in a nonsingular matrix is a linearly independent set, and because any subset of an independent set is again independent—see (4.3.7) and (4.3.14)—it follows that the last $m-r$ rows in $\mathbf{P}$ are linearly independent, and hence they constitute a basis for $N\left(\mathbf{A}^T\right)$. And this implies $\dim N\left(\mathbf{A}^T\right) = m - r$ (i.e., the number of rows in $\mathbf{A}$ minus the rank of $\mathbf{A}$). Replacing $\mathbf{A}$ by $\mathbf{A}^T$ shows that $\dim N\left(\mathbf{A}^{T^T}\right) = \dim N(\mathbf{A})$ is the number of rows in $\mathbf{A}^T$ minus $rank\left(\mathbf{A}^T\right)$. But $rank\left(\mathbf{A}^T\right) = rank(\mathbf{A}) = r$, so $\dim N(\mathbf{A}) = n-r$. We deduced $\dim N(\mathbf{A})$ without exhibiting a specific basis, but a basis for $N(\mathbf{A})$ is easy to describe. Recall that the set $\mathcal{H}$ containing the $\mathbf{h}_i$'s appearing in the general solution (4.2.9) of $\mathbf{Ax} = \mathbf{0}$ spans $N(\mathbf{A})$. Since there are exactly $n-r$ vectors in $\mathcal{H}$, and since $\dim N(\mathbf{A}) = n - r$, $\mathcal{H}$ is a minimal spanning set, so, by (4.4.2), $\mathcal{H}$ must be a basis for $N(\mathbf{A})$. Below is a summary of facts uncovered above.

Fundamental Subspaces—Dimension and Bases

For an $m \times n$ matrix of real numbers such that $rank(\mathbf{A}) = r$,

- $\dim R(\mathbf{A}) = r,$ (4.4.7)
- $\dim N(\mathbf{A}) = n - r,$ (4.4.8)
- $\dim R\left(\mathbf{A}^T\right) = r,$ (4.4.9)
- $\dim N\left(\mathbf{A}^T\right) = m - r.$ (4.4.10)

Let $\mathbf{P}$ be a nonsingular matrix such that $\mathbf{PA} = \mathbf{U}$ is in row echelon form, and let $\mathcal{H}$ be the set of $\mathbf{h}_i$'s appearing in the general solution (4.2.9) of $\mathbf{Ax} = \mathbf{0}$.

- The basic columns of $\mathbf{A}$ form a basis for $R(\mathbf{A})$. (4.4.11)
- The nonzero rows of $\mathbf{U}$ form a basis for $R\left(\mathbf{A}^T\right)$. (4.4.12)
- The set $\mathcal{H}$ is a basis for $N(\mathbf{A})$. (4.4.13)
- The last $m-r$ rows of $\mathbf{P}$ form a basis for $N\left(\mathbf{A}^T\right)$. (4.4.14)

For matrices with complex entries, the above statements remain valid provided that $\mathbf{A}^T$ is replaced with $\mathbf{A}^*$.

Statements (4.4.7) and (4.4.8) combine to produce the following theorem.

Rank Plus Nullity Theorem

- $\dim R(\mathbf{A}) + \dim N(\mathbf{A}) = n$ for all $m \times n$ matrices. (4.4.15)

In loose terms, this is a kind of conservation law—it says that as the amount of "stuff" in $R(\mathbf{A})$ increases, the amount of "stuff" in $N(\mathbf{A})$ must decrease, and vice versa. The phrase *rank plus nullity* is used because $\dim R(\mathbf{A})$ is the rank of $\mathbf{A}$, and $\dim N(\mathbf{A})$ was traditionally known as the ***nullity of*** $\mathbf{A}$.

Example 4.4.4

Problem: Determine the dimension as well as a basis for the space spanned by

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \right\}.$$

Solution 1: Place the vectors as columns in a matrix $\mathbf{A}$, and reduce

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{pmatrix} \longrightarrow \mathbf{E_A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $span(\mathcal{S}) = R(\mathbf{A})$, we have

$$\dim \big(span(\mathcal{S})\big) = \dim R(\mathbf{A}) = rank(\mathbf{A}) = 2.$$

The basic columns $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$ are a basis for $R(\mathbf{A}) = span(\mathcal{S})$. Other bases are also possible. Examining $\mathbf{E_A}$ reveals that any two vectors in $\mathcal{S}$ form an independent set, and therefore any pair of vectors from $\mathcal{S}$ constitutes a basis for $span(\mathcal{S})$.

Solution 2: Place the vectors from $\mathcal{S}$ as rows in a matrix $\mathbf{B}$, and reduce $\mathbf{B}$ to row echelon form:

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 5 & 6 & 7 \end{pmatrix} \longrightarrow \mathbf{U} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This time we have $span(\mathcal{S}) = R(\mathbf{B}^T)$, so that

$$\dim \big(span(\mathcal{S})\big) = \dim R(\mathbf{B}^T) = rank(\mathbf{B}) = rank(\mathbf{U}) = 2,$$

and a basis for $span(\mathcal{S}) = R(\mathbf{B}^T)$ is given by the nonzero rows in $\mathbf{U}$.

Example 4.4.5

Problem: If $\mathcal{S}_r = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent subset of an n-dimensional space $\mathcal{V}$, where $r < n$, explain why it must be possible to find extension vectors $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ from $\mathcal{V}$ such that

$$\mathcal{S}_n = \{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$$

is a basis for $\mathcal{V}$.

Solution 1: $r < n$ means that $span(\mathcal{S}_r) \neq \mathcal{V}$, and hence there exists a vector $\mathbf{v}_{r+1} \in \mathcal{V}$ such that $\mathbf{v}_{r+1} \notin span(\mathcal{S}_r)$. The extension set $\mathcal{S}_{r+1} = \mathcal{S}_r \cup \{\mathbf{v}_{r+1}\}$ is an independent subset of $\mathcal{V}$ containing $r+1$ vectors—recall (4.3.15). Repeating this process generates independent subsets $\mathcal{S}_{r+2}, \mathcal{S}_{r+3}, \ldots,$ and eventually leads to a maximal independent subset $\mathcal{S}_n \subset \mathcal{V}$ containing n vectors.

Solution 2: The first solution shows that it is theoretically possible to find extension vectors, but the argument given is not much help in actually computing them. It is easy to remedy this situation. Let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be any basis for $\mathcal{V}$, and place the given $\mathbf{v}_i$'s along with the $\mathbf{b}_i$'s as columns in a matrix

$$\mathbf{A} = \big(\mathbf{v}_1 \,|\, \cdots \,|\, \mathbf{v}_r \,|\, \mathbf{b}_1 \,|\, \cdots \,|\, \mathbf{b}_n\big).$$

Clearly, $R(\mathbf{A}) = \mathcal{V}$ so that the set of basic columns from $\mathbf{A}$ is a basis for $\mathcal{V}$. Observe that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ are basic columns in $\mathbf{A}$ because no one of these is a combination of preceding ones. Therefore, the remaining $n-r$ basic columns must be a subset of $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$—say they are $\left\{\mathbf{b}_{j_1}, \mathbf{b}_{j_2}, \ldots, \mathbf{b}_{j_{n-r}}\right\}$. The complete set of basic columns from $\mathbf{A}$, and a basis for $\mathcal{V}$, is the set

$$\mathcal{B} = \left\{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{b}_{j_1}, \ldots, \mathbf{b}_{j_{n-r}}\right\}.$$

For example, to extend the independent set

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix} \right\}$$

to a basis for $\Re^4$, append the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ to the vectors in $\mathcal{S}$, and perform the reduction

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & -2 & 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \mathbf{E_A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1/2 \end{pmatrix}.$$

This reveals that $\{\mathbf{A}_{*1}, \mathbf{A}_{*2}, \mathbf{A}_{*4}, \mathbf{A}_{*5}\}$ are the basic columns in $\mathbf{A}$, and therefore

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for $\Re^4$ that contains $\mathcal{S}$.

Example 4.4.6

Rank and Connectivity. A set of points (or *nodes*), $\{N_1, N_2, \ldots, N_m\}$, together with a set of paths (or *edges*), $\{E_1, E_2, \ldots, E_n\}$, between the nodes is called a ***graph***. A *connected graph* is one in which there is a sequence of edges linking any pair of nodes, and a *directed graph* is one in which each edge has been assigned a direction. For example, the graph in Figure 4.4.1 is both connected and directed.

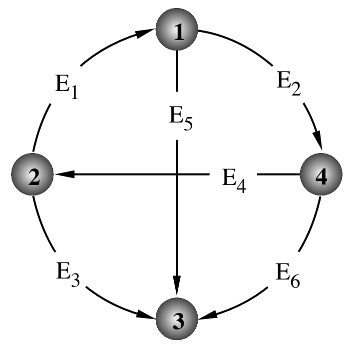


FIGURE 4.4.1

The connectivity of a directed graph is independent of the directions assigned to the edges—i.e., changing the direction of an edge doesn't change the connectivity. (Exercise 4.4.20 presents another type of connectivity in which direction matters.) On the surface, the concepts of graph connectivity and matrix rank seem to have little to do with each other, but, in fact, there is a close relationship. The ***incidence matrix*** associated with a directed graph containing m nodes and n edges is defined to be the $m \times n$ matrix $\mathbf{E}$ whose (k, j)-entry is

$$e_{kj} = \begin{cases} 1 & \text{if edge } E_j \text{ is directed } \textit{toward} \text{ node } N_k. \\ -1 & \text{if edge } E_j \text{ is directed } \textit{away from} \text{ node } N_k. \\ 0 & \text{if edge } E_j \text{ neither begins nor ends at node } N_k. \end{cases}$$

For example, the incidence matrix associated with the graph in Figure 4.4.1 is

$$\mathbf{E} = \begin{array}{c} \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & -1 \end{pmatrix} \end{array}. \tag{4.4.16}$$

Each edge in a directed graph is associated with two nodes—the nose and the tail of the edge—so each column in $\mathbf{E}$ must contain exactly two nonzero entries—a $(+1)$ and a (-1). Consequently, all column sums are zero. In other words, if $\mathbf{e}^T = (1 \quad 1 \quad \cdots \quad 1)$, then $\mathbf{e}^T\mathbf{E} = \mathbf{0}$, so $\mathbf{e} \in N(\mathbf{E}^T)$, and

$$rank(\mathbf{E}) = rank\left(\mathbf{E}^T\right) = m - \dim N\left(\mathbf{E}^T\right) \le m - 1. \tag{4.4.17}$$

This inequality holds regardless of the connectivity of the associated graph, but marvelously, equality is attained if and only if the graph is connected.

Rank and Connectivity

Let $\mathcal{G}$ be a graph containing m nodes. If $\mathcal{G}$ is undirected, arbitrarily assign directions to the edges to make $\mathcal{G}$ directed, and let $\mathbf{E}$ be the corresponding incidence matrix.

- $\mathcal{G}$ is connected if and only if $rank\,(\mathbf{E}) = m-1.$ (4.4.18)

Proof. Suppose $\mathcal{G}$ is connected. Prove $rank\,(\mathbf{E}) = m-1$ by arguing that $\dim N\left(\mathbf{E}^T\right) = 1,$ and do so by showing $\mathbf{e} = (\,1 \quad 1 \quad \cdots \quad 1\,)^T$ is a basis $N\left(\mathbf{E}^T\right)$. To see that $\mathbf{e}$ spans $N\left(\mathbf{E}^T\right)$, consider an arbitrary $\mathbf{x} \in N\left(\mathbf{E}^T\right)$, and focus on any two components x_i and x_k in $\mathbf{x}$ along with the corresponding nodes N_i and N_k in $\mathcal{G}$. Since $\mathcal{G}$ is connected, there must exist a subset of r nodes,

$$\{N_{j_1}, N_{j_2}, \ldots, N_{j_r}\}\,, \quad \text{where} \quad i = j_1 \quad \text{and} \quad k = j_r,$$

such that there is an edge between N_{j_p} and $N_{j_{p+1}}$ for each $p = 1, 2, \ldots, r-1$. Therefore, corresponding to each of the $r-1$ pairs $\left(N_{j_p}, N_{j_{p+1}}\right)$, there must exist a column $\mathbf{c}_p$ in $\mathbf{E}$ (not necessarily the p^{th} column) such that components j_p and j_{p+1} in $\mathbf{c}_p$ are complementary in the sense that one is $(+1)$ while the other is (-1) (all other components are zero). Because $\mathbf{x}^T\mathbf{E} = \mathbf{0}$, it follows that $\mathbf{x}^T\mathbf{c}_p = 0$, and hence $x_{j_p} = x_{j_{p+1}}$. But this holds for every $p = 1, 2, \ldots, r-1$, so $x_i = x_k$ for each i and k, and hence $\mathbf{x} = \alpha\mathbf{e}$ for some scalar α. Thus $\{\mathbf{e}\}$ spans $N\left(\mathbf{E}^T\right)$. Clearly, $\{\mathbf{e}\}$ is linearly independent, so it is a basis $N\left(\mathbf{E}^T\right)$, and, therefore, $\dim N\left(\mathbf{E}^T\right) = 1$ or, equivalently, $rank\,(\mathbf{E}) = m-1$. Conversely, suppose $rank\,(\mathbf{E}) = m-1$, and prove $\mathcal{G}$ is connected with an indirect argument. If $\mathcal{G}$ is not connected, then $\mathcal{G}$ is decomposable into two nonempty subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in which there are no edges between nodes in $\mathcal{G}_1$ and nodes in $\mathcal{G}_2$. This means that the nodes in $\mathcal{G}$ can be ordered so as to make $\mathbf{E}$ have the form

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_2 \end{pmatrix},$$

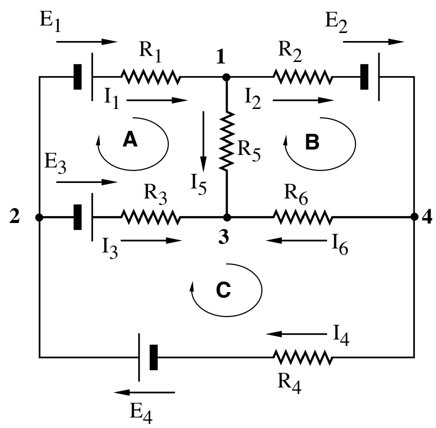
where $\mathbf{E}_1$ and $\mathbf{E}_2$ are the incidence matrices for $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. If $\mathcal{G}_1$ and $\mathcal{G}_2$ contain m_1 and m_2 nodes, respectively, then (4.4.17) insures that

$$rank\,(\mathbf{E}) = rank\begin{pmatrix} \mathbf{E}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_2 \end{pmatrix} = rank\,(\mathbf{E}_1) + rank\,(\mathbf{E}_1) \le (m_1 - 1) + (m_2 - 1) = m-2.$$

But this contradicts the hypothesis that $rank\,(\mathbf{E}) = m-1$, so the supposition that $\mathcal{G}$ is not connected must be false. ■

Example 4.4.7

An Application to Electrical Circuits. Recall from the discussion on p. 73 that applying Kirchhoff's node rule to an electrical circuit containing m nodes and n branches produces m homogeneous linear equations in n unknowns (the branch currents), and Kirchhoff's loop rule provides a nonhomogeneous equation for each simple loop in the circuit. For example, consider the circuit in Figure 4.4.2 along with its four nodal equations and three loop equations—this is the same circuit appearing on p. 73, and the equations are derived there.



$$\begin{aligned}
\text{Node 1:}\quad & I_1 - I_2 - I_5 = 0\\
\text{Node 2:}\quad & -I_1 - I_3 + I_4 = 0\\
\text{Node 3:}\quad & I_3 + I_5 + I_6 = 0\\
\text{Node 4:}\quad & I_2 - I_4 - I_6 = 0\\
\text{Loop A:}\quad & I_1R_1 - I_3R_3 + I_5R_5 = E_1 - E_3\\
\text{Loop B:}\quad & I_2R_2 - I_5R_5 + I_6R_6 = E_2\\
\text{Loop C:}\quad & I_3R_3 + I_4R_4 - I_6R_6 = E_3 + E_4
\end{aligned}$$

FIGURE 4.4.2

The directed graph and associated incidence matrix $\mathbf{E}$ defined by this circuit are the same as those appearing in Example 4.4.6 in Figure 4.4.1 and equation (4.4.16), so it's apparent that the 4×3 homogeneous system of nodal equations is precisely the system $\mathbf{Ex} = \mathbf{0}$. This observation holds for general circuits. The goal is to compute the six currents $I_1, I_2, \ldots, I_6$ by selecting six independent equations from the entire set of node and loop equations. In general, if a circuit containing m nodes is connected in the graph sense, then (4.4.18) insures that $rank\,(\mathbf{E}) = m - 1$, so there are m independent nodal equations. But Example 4.4.6 also shows that $\mathbf{0} = \mathbf{e}^T\mathbf{E} = \mathbf{E}_{1*} + \mathbf{E}_{2*} + \cdots + \mathbf{E}_{m*}$, which means that any row can be written in terms of the others, and this in turn implies that *every* subset of $m-1$ rows in $\mathbf{E}$ must be independent (see Exercise 4.4.13). Consequently, when any nodal equation is discarded, the remaining ones are guaranteed to be independent. To determine an $n \times n$ nonsingular system that has the n branch currents as its unique solution, it's therefore necessary to find $n-m+1$ additional independent equations, and, as shown in §2.6, these are the loop equations. A simple loop in a circuit is now seen to be a connected subgraph that does not properly contain other connected subgraphs. Physics dictates that the currents must be uniquely determined, so there must always be $n-m+1$ simple loops, and the combination of these loop equations together with any subset of $m-1$ nodal equations will be a nonsingular $n \times n$ system that yields the branch currents as its unique solution. For example, any three of the nodal equations in Figure 4.4.2 can be coupled with the three simple loop equations to produce a 6×6 nonsingular system whose solution is the six branch currents.

If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$, then the sum of $\mathcal{X}$ and $\mathcal{Y}$ was defined in §4.1 to be

$$\mathcal{X}+\mathcal{Y}=\{\mathbf{x}+\mathbf{y} \mid \mathbf{x}\in\mathcal{X} \text{ and } \mathbf{y}\in\mathcal{Y}\},$$

and it was demonstrated in (4.1.1) that $\mathcal{X}+\mathcal{Y}$ is again a subspace of $\mathcal{V}$. You were asked in Exercise 4.1.8 to prove that the intersection $\mathcal{X}\cap\mathcal{Y}$ is also a subspace of $\mathcal{V}$. We are now in a position to exhibit an important relationship between $\dim(\mathcal{X}+\mathcal{Y})$ and $\dim(\mathcal{X}\cap\mathcal{Y})$.

Dimension of a Sum

If $\mathcal{X}$ and $\mathcal{Y}$ are subspaces of a vector space $\mathcal{V}$, then

$$\dim(\mathcal{X}+\mathcal{Y})=\dim\mathcal{X}+\dim\mathcal{Y}-\dim(\mathcal{X}\cap\mathcal{Y}). \tag{4.4.19}$$

Proof. The strategy is to construct a basis for $\mathcal{X}+\mathcal{Y}$ and count the number of vectors it contains. Let $\mathcal{S}=\{\mathbf{z}_1,\mathbf{z}_2,\ldots,\mathbf{z}_t\}$ be a basis for $\mathcal{X}\cap\mathcal{Y}$. Since $\mathcal{S}\subseteq\mathcal{X}$ and $\mathcal{S}\subseteq\mathcal{Y}$, there must exist extension vectors $\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m\}$ and $\{\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n\}$ such that

$$\mathcal{B}_X=\{\mathbf{z}_1,\ldots,\mathbf{z}_t,\mathbf{x}_1,\ldots,\mathbf{x}_m\}= \text{ a basis for } \mathcal{X}$$

and

$$\mathcal{B}_Y=\{\mathbf{z}_1,\ldots,\mathbf{z}_t,\mathbf{y}_1,\ldots,\mathbf{y}_n\}= \text{ a basis for } \mathcal{Y}.$$

We know from (4.1.2) that $\mathcal{B}=\mathcal{B}_X\cup\mathcal{B}_Y$ spans $\mathcal{X}+\mathcal{Y}$, and we wish show that $\mathcal{B}$ is linearly independent. If

$$\sum_{i=1}^{t}\alpha_i\mathbf{z}_i+\sum_{j=1}^{m}\beta_j\mathbf{x}_j+\sum_{k=1}^{n}\gamma_k\mathbf{y}_k=\mathbf{0}, \tag{4.4.20}$$

then

$$\sum_{k=1}^{n}\gamma_k\mathbf{y}_k=-\left(\sum_{i=1}^{t}\alpha_i\mathbf{z}_i+\sum_{j=1}^{m}\beta_j\mathbf{x}_j\right)\in\mathcal{X}.$$

Since it is also true that $\sum_k\gamma_k\mathbf{y}_k\in\mathcal{Y}$, we have that $\sum_k\gamma_k\mathbf{y}_k\in\mathcal{X}\cap\mathcal{Y}$, and hence there must exist scalars δ_i such that

$$\sum_{k=1}^{n}\gamma_k\mathbf{y}_k=\sum_{i=1}^{t}\delta_i\mathbf{z}_i \quad \text{or, equivalently,} \quad \sum_{k=1}^{n}\gamma_k\mathbf{y}_k-\sum_{i=1}^{t}\delta_i\mathbf{z}_i=\mathbf{0}.$$

Since $\mathcal{B}_Y$ is an independent set, it follows that all of the γ_k 's (as well as all δ_i 's) are zero, and (4.4.20) reduces to $\sum_{i=1}^{t} \alpha_i \mathbf{z}_i + \sum_{j=1}^{m} \beta_j \mathbf{x}_j = \mathbf{0}$. But $\mathcal{B}_X$ is also an independent set, so the only way this can hold is for all of the α_i 's as well as all of the β_j 's to be zero. Therefore, the only possible solution for the α 's, β 's, and γ 's in the homogeneous equation (4.4.20) is the trivial solution, and thus $\mathcal{B}$ is linearly independent. Since $\mathcal{B}$ is an independent spanning set, it is a basis for $\mathcal{X} + \mathcal{Y}$ and, consequently,

$$\dim(\mathcal{X} + \mathcal{Y}) = t+m+n = (t+m)+(t+n)-t = \dim \mathcal{X} + \dim \mathcal{Y} - \dim(\mathcal{X} \cap \mathcal{Y}). \quad \blacksquare$$

Example 4.4.8

Problem: Show that $rank(\mathbf{A}+\mathbf{B}) \le rank(\mathbf{A}) + rank(\mathbf{B})$.

Solution: Observe that

$$R(\mathbf{A}+\mathbf{B}) \subseteq R(\mathbf{A}) + R(\mathbf{B})$$

because if $\mathbf{b} \in R(\mathbf{A}+\mathbf{B})$, then there is a vector $\mathbf{x}$ such that

$$\mathbf{b} = (\mathbf{A}+\mathbf{B})\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x} \in R(\mathbf{A}) + R(\mathbf{B}).$$

Recall from (4.4.5) that if $\mathcal{M}$ and $\mathcal{N}$ are vector spaces such that $\mathcal{M} \subseteq \mathcal{N}$, then $\dim \mathcal{M} \le \dim \mathcal{N}$. Use this together with formula (4.4.19) for the dimension of a sum to conclude that

$$\begin{aligned} rank(\mathbf{A}+\mathbf{B}) &= \dim R(\mathbf{A}+\mathbf{B}) \le \dim \Big(R(\mathbf{A}) + R(\mathbf{B})\Big) \\ &= \dim R(\mathbf{A}) + \dim R(\mathbf{B}) - \dim \Big(R(\mathbf{A}) \cap R(\mathbf{B})\Big) \\ &\le \dim R(\mathbf{A}) + \dim R(\mathbf{B}) = rank(\mathbf{A}) + rank(\mathbf{B}). \end{aligned}$$

Exercises for section 4.4

4.4.1. Find the dimensions of the four fundamental subspaces associated with

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}.$$

4.4.2. Find a basis for each of the four fundamental subspaces associated with

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ 3 & 6 & 1 & 9 & 6 \\ 2 & 4 & 1 & 7 & 5 \end{pmatrix}.$$

4.4.3. Determine the dimension of the space spanned by the set

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \\ -4 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \\ 6 \end{pmatrix} \right\}.$$

4.4.4. Determine the dimensions of each of the following vector spaces:

(a) The space of polynomials having degree n or less.
(b) The space $\Re^{m\times n}$ of $m \times n$ matrices.
(c) The space of $n \times n$ symmetric matrices.

4.4.5. Consider the following matrix and column vector:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 0 & 5 \\ 2 & 4 & 3 & 1 & 8 \\ 3 & 6 & 1 & 5 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} -8 \\ 1 \\ 3 \\ 3 \\ 0 \end{pmatrix}.$$

Verify that $\mathbf{v} \in N(\mathbf{A})$, and then extend $\{\mathbf{v}\}$ to a basis for $N(\mathbf{A})$.

4.4.6. Determine whether or not the set

$$\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

is a basis for the space spanned by the set

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \right\}.$$

4.4.7. Construct a 4×4 homogeneous system of equations that has no zero coefficients and three linearly independent solutions.

4.4.8. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space $\mathcal{V}$. Prove that each $\mathbf{v} \in \mathcal{V}$ can be expressed as a linear combination of the $\mathbf{b}_i$'s

$$\mathbf{v} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \cdots + \alpha_n \mathbf{b}_n,$$

in only one way—i.e., the *coordinates* α_i are unique.

4.4.9. For $\mathbf{A} \in \Re^{m \times n}$ and a subspace $\mathcal{S}$ of $\Re^{n\times 1}$, the image

$$\mathbf{A}(\mathcal{S}) = \{\mathbf{Ax} \,|\, \mathbf{x} \in \mathcal{S}\}$$

of $\mathcal{S}$ under $\mathbf{A}$ is a subspace of $\Re^{m\times 1}$—recall Exercise 4.1.9. Prove that if $\mathcal{S} \cap N(\mathbf{A}) = \mathbf{0}$, then $\dim \mathbf{A}(\mathcal{S}) = \dim(\mathcal{S})$. **Hint:** Use a basis $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ for $\mathcal{S}$ to determine a basis for $\mathbf{A}(\mathcal{S})$.

4.4.10. Explain why $\big|rank(\mathbf{A}) - rank(\mathbf{B})\big| \le rank(\mathbf{A} - \mathbf{B})$.

4.4.11. If $rank(\mathbf{A}_{m\times n}) = r$ and $rank(\mathbf{E}_{m\times n}) = k \le r$, explain why

$$r - k \le rank(\mathbf{A} + \mathbf{E}) \le r + k.$$

In words, this says that a perturbation of rank k can change the rank by at most k.

4.4.12. Explain why every nonzero subspace $\mathcal{V} \subseteq \Re^n$ must possess a basis.

4.4.13. Explain why *every* set of $m-1$ rows in the incidence matrix $\mathbf{E}$ of a connected directed graph containing m nodes is linearly independent.

4.4.14. For the incidence matrix $\mathbf{E}$ of a directed graph, explain why

$$\left[\mathbf{E}\mathbf{E}^T\right]_{ij} = \begin{cases} \text{number of edges at node } i & \text{when } i = j, \\ -(\text{number of edges between nodes } i \text{ and } j) & \text{when } i \ne j. \end{cases}$$

4.4.15. If $\mathcal{M}$ and $\mathcal{N}$ are subsets of a space $\mathcal{V}$, explain why

$$\begin{aligned} \dim\big(span(\mathcal{M}\cup\mathcal{N})\big) &= \dim\big(span(\mathcal{M})\big) + \dim\big(span(\mathcal{N})\big) \\ &\quad - \dim\big(span(\mathcal{M}) \cap span(\mathcal{N})\big). \end{aligned}$$

4.4.16. Consider two matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{m\times k}$.

(a) Explain why

$$rank(\mathbf{A} \mid \mathbf{B}) = rank(\mathbf{A}) + rank(\mathbf{B}) - \dim\big(R(\mathbf{A}) \cap R(\mathbf{B})\big).$$

Hint: Recall Exercise 4.2.9.

(b) Now explain why

$$\dim N(\mathbf{A} \mid \mathbf{B}) = \dim N(\mathbf{A}) + \dim N(\mathbf{B}) + \dim\big(R(\mathbf{A}) \cap R(\mathbf{B})\big).$$

(c) Determine $\dim\big(R(\mathbf{C}) \cap N(\mathbf{C})\big)$ and $\dim\big(R(\mathbf{C}) + N(\mathbf{C})\big)$ for

$$\mathbf{C} = \begin{pmatrix} -1 & 1 & 1 & -2 & 1 \\ -1 & 0 & 3 & -4 & 2 \\ -1 & 0 & 3 & -5 & 3 \\ -1 & 0 & 3 & -6 & 4 \\ -1 & 0 & 3 & -6 & 4 \end{pmatrix}.$$

4.4.17. Suppose that $\mathbf{A}$ is a matrix with m rows such that the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \Re^m$. Explain why this means that $\mathbf{A}$ must be square and nonsingular.

4.4.18. Let $\mathcal{S}$ be the solution set for a consistent system of linear equations $\mathbf{Ax} = \mathbf{b}$.

(a) If $\mathcal{S}_{max} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_t\}$ is a maximal independent subset of $\mathcal{S}$, and if $\mathbf{p}$ is any particular solution, prove that

$$span\,(\mathcal{S}_{max}) = span\,\{\mathbf{p}\} + N\,(\mathbf{A}).$$

Hint: First show that $\mathbf{x} \in \mathcal{S}$ implies $\mathbf{x} \in span\,(\mathcal{S}_{max})$, and then demonstrate set inclusion in both directions with the aid of Exercise 4.2.10.

(b) If $\mathbf{b} \neq \mathbf{0}$ and $rank\,(\mathbf{A}_{m\times n}) = r$, explain why $\mathbf{Ax} = \mathbf{b}$ has $n - r + 1$ "independent solutions."

4.4.19. Let $rank\,(\mathbf{A}_{m\times n}) = r$, and suppose $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{b} \neq \mathbf{0}$ is a consistent system. If $\mathcal{H} = \{\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_{n-r}\}$ is a basis for $N\,(\mathbf{A})$, and if $\mathbf{p}$ is a particular solution to $\mathbf{Ax} = \mathbf{b}$, show that

$$\mathcal{S}_{max} = \{\mathbf{p},\ \mathbf{p} + \mathbf{h}_1,\ \mathbf{p} + \mathbf{h}_2,\ \ldots,\ \mathbf{p} + \mathbf{h}_{n-r}\}$$

is a maximal independent set of solutions.

4.4.20. Strongly Connected Graphs. In Example 4.4.6 we started with a graph to construct a matrix, but it's also possible to reverse the situation by starting with a matrix to build an associated graph. The graph of $\mathbf{A}_{n\times n}$ (denoted by $\mathcal{G}(\mathbf{A})$) is defined to be the directed graph on n nodes $\{N_1, N_2, \ldots, N_n\}$ in which there is a directed edge leading *from* N_i *to* N_j if and only if $a_{ij} \neq 0$. The directed graph $\mathcal{G}(\mathbf{A})$ is said to be ***strongly connected*** provided that for each pair of nodes (N_i, N_k) there is a sequence of directed edges leading *from* N_i *to* N_k. The matrix $\mathbf{A}$ is said to be ***reducible*** if there exists a permutation matrix $\mathbf{P}$ such that $\mathbf{P}^T\mathbf{AP} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix}$, where $\mathbf{X}$ and $\mathbf{Z}$ are both square matrices. Otherwise, $\mathbf{A}$ is said to be ***irreducible.*** Prove that $\mathcal{G}(\mathbf{A})$ is strongly connected if and only if $\mathbf{A}$ is irreducible. **Hint:** Prove the contrapositive: $\mathcal{G}(\mathbf{A})$ is *not* strongly connected if and only if $\mathbf{A}$ is reducible.

Solutions for exercises in section 4. 4

4.4.1. $\dim R(\mathbf{A}) = \dim R\left(\mathbf{A}^T\right) = rank(\mathbf{A}) = 2, \quad \dim N(\mathbf{A}) = n - r = 4 - 2 = 2,$ and $\dim N\left(\mathbf{A}^T\right) = m - r = 3 - 2 = 1.$

4.4.2. $\mathcal{B}_{R(\mathbf{A})} = \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{B}_{N(\mathbf{A}^T)} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

$$\mathcal{B}_{R(\mathbf{A}^T)} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 3 \end{pmatrix} \right\}, \quad \mathcal{B}_{N(\mathbf{A})} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

4.4.3. $\dim\left(span(\mathcal{S})\right) = 3$

4.4.4. (a) $n+1$ (See Example 4.4.1) (b) mn (c) $\frac{n^2+n}{2}$

4.4.5. Use the technique of Example 4.4.5. Find $\mathbf{E_A}$ to determine

$$\left\{ \mathbf{h}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{h}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{h}_3 = \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $N(\mathbf{A})$. Reducing the matrix $\left(\mathbf{v}, \ \mathbf{h}_1, \ \mathbf{h}_2, \ \mathbf{h}_3\right)$ to row echelon form reveals that its first, second, and fourth columns are basic, and hence $\{\mathbf{v}, \ \mathbf{h}_1, \ \mathbf{h}_3\}$ is a basis for $N(\mathbf{A})$ that contains $\mathbf{v}$.

4.4.6. Placing the vectors from $\mathcal{A}$ and $\mathcal{B}$ as rows in matrices and reducing

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 8 & 7 \\ 3 & 4 & 1 \end{pmatrix} \longrightarrow \mathbf{E_A} = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & -1 \end{pmatrix} \longrightarrow \mathbf{E_B} = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \end{pmatrix}$$

shows $\mathbf{A}$ and $\mathbf{B}$ have the same row space (recall Example 4.2.2), and hence $\mathcal{A}$ and $\mathcal{B}$ span the same space. Because $\mathcal{B}$ is linearly independent, it follows that $\mathcal{B}$ is a basis for $span(\mathcal{A})$.

4.4.7. $3 = \dim N(\mathbf{A}) = n - r = 4 - rank(\mathbf{A}) \implies rank(\mathbf{A}) = 1.$ Therefore, any rank-one matrix with no zero entries will do the job.

4.4.8. If $\mathbf{v} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \cdots + \alpha_n\mathbf{b}_n$ and $\mathbf{v} = \beta_1\mathbf{b}_1 + \beta_2\mathbf{b}_2 + \cdots + \beta_n\mathbf{b}_n,$ then subtraction produces

$$\mathbf{0} = (\alpha_1 - \beta_1)\mathbf{b}_1 + (\alpha_2 - \beta_2)\mathbf{b}_2 + \cdots + (\alpha_n - \beta_n)\mathbf{b}_n.$$

But $\mathcal{B}$ is a linearly independent set, so this equality can hold only if $(\alpha_i - \beta_i) = 0$ for each $i = 1, 2, \ldots, n$, and hence the α_i's are unique.

4.4.9. Prove that if $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ is a basis for $\mathcal{S}$, then $\{\mathbf{As}_1, \mathbf{As}_2, \ldots, \mathbf{As}_k\}$ is a basis for $\mathbf{A}(\mathcal{S})$. The result of Exercise 4.1.9 insures that

$$\mathit{span}\,\{\mathbf{As}_1, \mathbf{As}_2, \ldots, \mathbf{As}_k\} = \mathbf{A}(\mathcal{S}),$$

so we need only establish the independence of $\{\mathbf{As}_1, \mathbf{As}_2, \ldots, \mathbf{As}_k\}$. To do this, write

$$\begin{aligned}
\sum_{i=1}^{k} \alpha_i \left(\mathbf{As}_i\right) = \mathbf{0} \quad &\Longrightarrow \quad \mathbf{A}\left(\sum_{i=1}^{k} \alpha_i \mathbf{s}_i\right) = \mathbf{0} \quad \Longrightarrow \quad \sum_{i=1}^{k} \alpha_i \mathbf{s}_i \in N\left(\mathbf{A}\right) \\
&\Longrightarrow \quad \sum_{i=1}^{k} \alpha_i \mathbf{s}_i = \mathbf{0} \quad \text{because} \quad \mathcal{S} \cap N\left(\mathbf{A}\right) = \mathbf{0} \\
&\Longrightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0
\end{aligned}$$

because $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k\}$ is linearly independent. Since $\{\mathbf{As}_1, \mathbf{As}_2, \ldots, \mathbf{As}_k\}$ is a basis for $\mathbf{A}(\mathcal{S})$, it follows that $\dim \mathbf{A}(\mathcal{S}) = k = \dim(\mathcal{S})$.

4.4.10. $\mathit{rank}\left(\mathbf{A}\right) = \mathit{rank}\left(\mathbf{A} - \mathbf{B} + \mathbf{B}\right) \le \mathit{rank}\left(\mathbf{A} - \mathbf{B}\right) + \mathit{rank}\left(\mathbf{B}\right)$ implies that

$$\mathit{rank}\left(\mathbf{A}\right) - \mathit{rank}\left(\mathbf{B}\right) \le \mathit{rank}\left(\mathbf{A} - \mathbf{B}\right).$$

Furthermore, $\mathit{rank}\left(\mathbf{B}\right) = \mathit{rank}\left(\mathbf{B} - \mathbf{A} + \mathbf{A}\right) \le \mathit{rank}\left(\mathbf{B} - \mathbf{A}\right) + \mathit{rank}\left(\mathbf{A}\right) = \mathit{rank}\left(\mathbf{A} - \mathbf{B}\right) + \mathit{rank}\left(\mathbf{A}\right)$ implies that

$$-\Big(\mathit{rank}\left(\mathbf{A}\right) - \mathit{rank}\left(\mathbf{B}\right)\Big) \le \mathit{rank}\left(\mathbf{A} - \mathbf{B}\right).$$

4.4.11. Example 4.4.8 guarantees that $\mathit{rank}\left(\mathbf{A} + \mathbf{E}\right) \le \mathit{rank}\left(\mathbf{A}\right) + \mathit{rank}\left(\mathbf{E}\right) = r + k$. Use Exercise 4.4.10 to write

$$\mathit{rank}\left(\mathbf{A} + \mathbf{E}\right) = \mathit{rank}\left(\mathbf{A} - (-\mathbf{E})\right) \ge \mathit{rank}\left(\mathbf{A}\right) - \mathit{rank}\left(-\mathbf{E}\right) = r - k.$$

4.4.12. Let $\mathbf{v}_1 \in \mathcal{V}$ such that $\mathbf{v}_1 \neq \mathbf{0}$. If $\mathit{span}\left\{\mathbf{v}_1\right\} = \mathcal{V}$, then $\mathcal{S}_1 = \{\mathbf{v}_1\}$ is an independent spanning set for $\mathcal{V}$, and we are finished. If $\mathit{span}\left\{\mathbf{v}_1\right\} \neq \mathcal{V}$, then there is a vector $\mathbf{v}_2 \in \mathcal{V}$ such that $\mathbf{v}_2 \notin \mathit{span}\left\{\mathbf{v}_1\right\}$, and hence the extension set $\mathcal{S}_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ is independent. If $\mathit{span}\left(\mathcal{S}_2\right) = \mathcal{V}$, then we are finished. Otherwise, we can proceed as described in Example 4.4.5 and continue to build independent extension sets $\mathcal{S}_3, \mathcal{S}_4, \ldots$. Statement (4.3.16) guarantees that the process must eventually yield a linearly independent spanning set $\mathcal{S}_k$ with $k \le n$.

4.4.13. Since $\mathbf{0} = \mathbf{e}^T\mathbf{E} = \mathbf{E}_{1*} + \mathbf{E}_{2*} + \cdots + \mathbf{E}_{m*}$, any row can be written as a combination of the other $m-1$ rows, so any set of $m-1$ rows from $\mathbf{E}$ spans $N\left(\mathbf{E}^T\right)$. Furthermore, $\mathit{rank}\left(\mathbf{E}\right) = m - 1$ insures that no fewer than $m-1$ vectors can span $N\left(\mathbf{E}^T\right)$, and therefore any set of $m-1$ rows from $\mathbf{E}$ is a minimal spanning set, and hence a basis.

4.4.14. $\left[\mathbf{E}\mathbf{E}^T\right]_{ij} = \mathbf{E}_{i*}\left(\mathbf{E}^T\right)_{*j} = \mathbf{E}_{i*}\left(\mathbf{E}_{j*}\right)^T = \sum_k e_{ik}e_{jk}$. Observe that edge E_k touches node N_i if and only if $e_{ik} = \pm 1$ or, equivalently, $e_{ik}^2 = 1$. Thus $\left[\mathbf{E}\mathbf{E}^T\right]_{ii} = \sum_k e_{ik}^2 =$ the number of edges touching N_i. If $i \neq j$, then

$$e_{ik}e_{jk} = \begin{cases} -1 & \text{if } E_k \text{ is between } N_i \text{ and } N_j \\ 0 & \text{if } E_k \text{ is not between } N_i \text{ and } N_j \end{cases}$$

so that $\left[\mathbf{E}\mathbf{E}^T\right]_{ij} = \sum_k e_{ik}e_{jk} = -$ (the number of edges between N_i and N_j).

4.4.15. Apply (4.4.19) to $span\left(\mathcal{M}\cup\mathcal{N}\right) = span\left(\mathcal{M}\right) + span\left(\mathcal{N}\right)$ (see Exercise 4.1.7).

4.4.16. (a) Exercise 4.2.9 says $R\left(\mathbf{A} \mid \mathbf{B}\right) = R\left(\mathbf{A}\right) + R\left(\mathbf{B}\right)$. Since rank is the same as dimension of the range, (4.4.19) yields

$$\begin{aligned} rank\left(\mathbf{A} \mid \mathbf{B}\right) &= \dim R\left(\mathbf{A} \mid \mathbf{B}\right) = \dim\Big(R\left(\mathbf{A}\right) + R\left(\mathbf{B}\right)\Big) \\ &= \dim R\left(\mathbf{A}\right) + \dim R\left(\mathbf{B}\right) - \dim\Big(R\left(\mathbf{A}\right)\cap R\left(\mathbf{B}\right)\Big) \\ &= rank\left(\mathbf{A}\right) + rank\left(\mathbf{B}\right) - \dim\Big(R\left(\mathbf{A}\right)\cap R\left(\mathbf{B}\right)\Big). \end{aligned}$$

(b) Use the results of part (a) to write

$$\begin{aligned} \dim N\left(\mathbf{A} \mid \mathbf{B}\right) &= n + k - rank\left(\mathbf{A} \mid \mathbf{B}\right) \\ &= \Big(n - rank\left(\mathbf{A}\right)\Big) + \Big(k - rank\left(\mathbf{B}\right)\Big) + \dim\Big(R\left(\mathbf{A}\right)\cap R\left(\mathbf{B}\right)\Big) \\ &= \dim N\left(\mathbf{A}\right) + \dim N\left(\mathbf{B}\right) + \dim\Big(R\left(\mathbf{A}\right)\cap R\left(\mathbf{B}\right)\Big). \end{aligned}$$

(c) Let $\mathbf{A} = \begin{pmatrix} -1 & 1 & -2 \\ -1 & 0 & -4 \\ -1 & 0 & -5 \\ -1 & 0 & -6 \\ -1 & 0 & -6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$ contain bases for $R\left(\mathbf{C}\right)$ and $N\left(\mathbf{C}\right)$, respectively, so that $R\left(\mathbf{A}\right) = R\left(\mathbf{C}\right)$ and $R\left(\mathbf{B}\right) = N\left(\mathbf{C}\right)$. Use either part (a) or part (b) to obtain

$$\dim\Big(R\left(\mathbf{C}\right)\cap N\left(\mathbf{C}\right)\Big) = \dim\Big(R\left(\mathbf{A}\right)\cap R\left(\mathbf{B}\right)\Big) = 2.$$

Using $R\left(\mathbf{A} \mid \mathbf{B}\right) = R\left(\mathbf{A}\right) + R\left(\mathbf{B}\right)$ produces

$$\dim\Big(R\left(\mathbf{C}\right) + N\left(\mathbf{C}\right)\Big) = \dim\Big(R\left(\mathbf{A}\right) + R\left(\mathbf{B}\right)\Big) = rank\left(\mathbf{A} \mid \mathbf{B}\right) = 3.$$

4.4.17. Suppose $\mathbf{A}$ is $m \times n$. Existence of a solution for every $\mathbf{b}$ implies $R\left(\mathbf{A}\right) = \Re^m$. Recall from §2.5 that uniqueness of the solution implies $rank\left(\mathbf{A}\right) = n$. Thus $m = \dim R\left(\mathbf{A}\right) = rank\left(\mathbf{A}\right) = n$ so that $\mathbf{A}$ is $m \times m$ of rank m.

4.4.18. (a) $\mathbf{x} \in \mathcal{S} \implies \mathbf{x} \in span(\mathcal{S}_{max})$—otherwise, the extension set $\mathcal{E} = \mathcal{S}_{max} \cup \{\mathbf{x}\}$ would be linearly independent—which is impossible because $\mathcal{E}$ would contain more independent solutions than $\mathcal{S}_{max}$. Now show $span(\mathcal{S}_{max}) \subseteq span\{\mathbf{p}\} + N(\mathbf{A})$. Since $\mathcal{S} = \mathbf{p} + N(\mathbf{A})$ (see Exercise 4.2.10), $\mathbf{s}_i \in \mathcal{S}$ means there must exist a corresponding vector $\mathbf{n}_i \in N(\mathbf{A})$ such that $\mathbf{s}_i = \mathbf{p} + \mathbf{n}_i$, and hence

$$\begin{aligned} \mathbf{x} \in span(\mathcal{S}_{max}) &\implies \mathbf{x} = \sum_{i=1}^{t} \alpha_i \mathbf{s}_i = \sum_{i=1}^{t} \alpha_i(\mathbf{p} + \mathbf{n}_i) = \sum_{i=1}^{t} \alpha_i \mathbf{p} + \sum_{i=1}^{t} \alpha_i \mathbf{n}_i \\ &\implies \mathbf{x} \in span\{\mathbf{p}\} + N(\mathbf{A}) \\ &\implies span(\mathcal{S}_{max}) \subseteq span\{\mathbf{p}\} + N(\mathbf{A}). \end{aligned}$$

To prove the reverse inclusion, observe that if $\mathbf{x} \in span\{\mathbf{p}\} + N(\mathbf{A})$, then there exists a scalar α and a vector $\mathbf{n} \in N(\mathbf{A})$ such that

$$\mathbf{x} = \alpha\mathbf{p} + \mathbf{n} = (\alpha - 1)\mathbf{p} + (\mathbf{p} + \mathbf{n}).$$

Because $\mathbf{p}$ and $(\mathbf{p} + \mathbf{n})$ are both solutions, $\mathcal{S} \subseteq span(\mathcal{S}_{\max})$ guarantees that $\mathbf{p}$ and $(\mathbf{p} + \mathbf{n})$ each belong to $span(\mathcal{S}_{max})$, and the closure properties of a subspace insure that $\mathbf{x} \in span(\mathcal{S}_{max})$. Thus $span\{\mathbf{p}\} + N(\mathbf{A}) \subseteq span(\mathcal{S}_{max})$.
(b) The problem is really to determine the value of t in $\mathcal{S}_{max}$. The fact that $\mathcal{S}_{max}$ is a basis for $span(\mathcal{S}_{max})$ together with (4.4.19) produces

$$\begin{aligned} t &= \dim\big(span(\mathcal{S}_{max})\big) = \dim\Big(span\{\mathbf{p}\} + N(\mathbf{A})\Big) \\ &= \dim\big(span\{\mathbf{p}\}\big) + \dim N(\mathbf{A}) - \dim\Big(span\{\mathbf{p}\} \cap N(\mathbf{A})\Big) \\ &= 1 + (n - r) - 0. \end{aligned}$$

4.4.19. To show $\mathcal{S}_{max}$ is linearly independent, suppose

$$\mathbf{0} = \alpha_0\mathbf{p} + \sum_{i=1}^{n-r} \alpha_i(\mathbf{p} + \mathbf{h}_i) = \left(\sum_{i=0}^{n-r} \alpha_i\right)\mathbf{p} + \sum_{i=1}^{n-r} \alpha_i \mathbf{h}_i.$$

Multiplication by $\mathbf{A}$ yields $\mathbf{0} = \left(\sum_{i=0}^{n-r} \alpha_i\right)\mathbf{b}$, which implies $\sum_{i=0}^{n-r} \alpha_i = 0$, and hence $\sum_{i=1}^{n-r} \alpha_i \mathbf{h}_i = \mathbf{0}$. Because $\mathcal{H}$ is independent, we may conclude that $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-r} = 0$. Consequently, $\alpha_0\mathbf{p} = \mathbf{0}$, and therefore $\alpha_0 = 0$ (because $\mathbf{p} \neq \mathbf{0}$), so that $\mathcal{S}_{max}$ is an independent set. By Exercise 4.4.18, it must also be maximal because it contains $n - r + 1$ vectors.

4.4.20. The proof depends on the observation that if $\mathbf{B} = \mathbf{P}^T\mathbf{A}\mathbf{P}$, where $\mathbf{P}$ is a permutation matrix, then the graph $\mathcal{G}(\mathbf{B})$ is the same as $\mathcal{G}(\mathbf{A})$ except that the nodes in $\mathcal{G}(\mathbf{B})$ have been renumbered according to the permutation defining $\mathbf{P}$. This follows because $\mathbf{P}^T = \mathbf{P}^{-1}$ implies $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^T$, so if the rows (and columns) in $\mathbf{P}$ are the unit vectors that appear according to the permutation $\boldsymbol{\pi} = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$, then

$$a_{ij} = \left[\mathbf{P}\mathbf{B}\mathbf{P}^T\right]_{ij} = \left[\begin{pmatrix} \mathbf{e}_{\pi_1}^T \\ \vdots \\ \mathbf{e}_{\pi_n}^T \end{pmatrix} \mathbf{B} \left(\mathbf{e}_{\pi_1} \quad \cdots \quad \mathbf{e}_{\pi_n} \right)\right]_{ij} = \mathbf{e}_{\pi_i}^T \mathbf{B} \mathbf{e}_{\pi_j} = b_{\pi_i \pi_j}.$$

Consequently, $a_{ij} \neq 0$ if and only if $b_{\pi_i\pi_j} \neq 0$, and thus $\mathcal{G}(\mathbf{A})$ and $\mathcal{G}(\mathbf{B})$ are the same except for the fact that node N_k in $\mathcal{G}(\mathbf{A})$ is node N_{π_k} in $\mathcal{G}(\mathbf{B})$ for each $k = 1, 2, \ldots, n$. Now we can prove $\mathcal{G}(\mathbf{A})$ is *not* strongly connected $\iff$ $\mathbf{A}$ is reducible. If $\mathbf{A}$ is reducible, then there is a permutation matrix such that $\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{B} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix}$, where $\mathbf{X}$ is $r \times r$ and $\mathbf{Z}$ is $n-r \times n-r$. The zero pattern in $\mathbf{B}$ indicates that the nodes $\{N_1, N_2, \ldots, N_r\}$ in $\mathcal{G}(\mathbf{B})$ are inaccessible from nodes $\{N_{r+1}, N_{r+2}, \ldots, N_n\}$, and hence $\mathcal{G}(\mathbf{B})$ is not strongly connected—e.g., there is no sequence of edges leading from N_{r+1} to N_1. Since $\mathcal{G}(\mathbf{B})$ is the same as $\mathcal{G}(\mathbf{A})$ except that the nodes have different numbers, we may conclude that $\mathcal{G}(\mathbf{A})$ is also not strongly connected. Conversely, if $\mathcal{G}(\mathbf{A})$ is not strongly connected, then there are two nodes in $\mathcal{G}(\mathbf{A})$ such that one is inaccessible from the other by any sequence of directed edges. Relabel the nodes in $\mathcal{G}(\mathbf{A})$ so that this pair is N_1 and N_n, where N_1 is inaccessible from N_n. If there are additional nodes—excluding N_n itself—which are also inaccessible from N_n, label them $N_2, N_3, \ldots, N_r$ so that the set of all nodes that are inaccessible from N_n—with the possible exception of N_n itself—is $\overline{N_n} = \{N_1, N_2, \ldots, N_r\}$ (inaccessible nodes). Label the remaining nodes—which are all accessible from N_n—as $\underline{N_n} = \{N_{r+1}, N_{r+2}, \ldots, N_{n-1}\}$ (accessible nodes). It follows that no node in $\overline{N_n}$ can be accessible from any node in $\underline{N_n}$, for otherwise nodes in $\overline{N_n}$ would be accessible from N_n through nodes in $\underline{N_n}$. In other words, if $N_{r+k} \in \underline{N_n}$ and $N_{r+k} \to N_i \in \overline{N_n}$, then $N_n \to N_{r+k} \to N_i$, which is impossible. This means that if $\boldsymbol{\pi} = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$ is the permutation generated by the relabeling process, then $a_{\pi_i\pi_j} = 0$ for each $i = r+1, r+2, \ldots, n-1$ and $j = 1, 2, \ldots, r$. Therefore, if $\mathbf{B} = \mathbf{P}^T\mathbf{A}\mathbf{P}$, where $\mathbf{P}$ is the permutation matrix corresponding to the permutation $\boldsymbol{\pi}$, then $b_{ij} = a_{\pi_i\pi_j}$, so $\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{B} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix}$, where $\mathbf{X}$ is $r \times r$ and $\mathbf{Z}$ is $n-r \times n-r$, and thus $\mathbf{A}$ is reducible.

Solutions for exercises in section 4. 5

4.5.1. $rank\left(\mathbf{A}^T\mathbf{A}\right) = rank\left(\mathbf{A}\right) = rank\left(\mathbf{A}\mathbf{A}^T\right) = 2$

4.5.2. $\dim N\left(\mathbf{A}\right) \cap R\left(\mathbf{B}\right) = rank\left(\mathbf{B}\right) - rank\left(\mathbf{A}\mathbf{B}\right) = 2 - 1 = 1.$