5.2 MATRIX NORMS

Because $\mathcal{C}^{m\times n}$ is a vector space of dimension mn, magnitudes of matrices $\mathbf{A} \in \mathcal{C}^{m\times n}$ can be "measured" by employing any vector norm on $\mathcal{C}^{mn}$. For example, by stringing out the entries of $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -4 & -2 \end{pmatrix}$ into a four-component vector, the euclidean norm on $\Re^4$ can be applied to write

$$\|\mathbf{A}\| = \left[2^2 + (-1)^2 + (-4)^2 + (-2)^2\right]^{1/2} = 5.$$

This is one of the simplest notions of a matrix norm, and it is called the *Frobenius* (p. 662) *norm* (older texts refer to it as the *Hilbert–Schmidt norm* or the *Schur norm*). There are several useful ways to describe the Frobenius matrix norm.

Frobenius Matrix Norm

The ***Frobenius norm*** of $\mathbf{A} \in \mathcal{C}^{m\times n}$ is defined by the equations

$$\|\mathbf{A}\|_F^2 = \sum_{i,j} |a_{ij}|^2 = \sum_i \|\mathbf{A}_{i*}\|_2^2 = \sum_j \|\mathbf{A}_{*j}\|_2^2 = trace\,(\mathbf{A}^*\mathbf{A}). \qquad (5.2.1)$$

The Frobenius matrix norm is fine for some problems, but it is not well suited for all applications. So, similar to the situation for vector norms, alternatives need to be explored. But before trying to develop different recipes for matrix norms, it makes sense to first formulate a general definition of a matrix norm. The goal is to start with the defining properties for a vector norm given in (5.1.9) on p. 275 and ask what, if anything, needs to be added to that list.

Matrix multiplication distinguishes matrix spaces from more general vector spaces, but the three vector-norm properties (5.1.9) say nothing about products. So, an extra property that relates $\|\mathbf{AB}\|$ to $\|\mathbf{A}\|$ and $\|\mathbf{B}\|$ is needed. The Frobenius norm suggests the nature of this extra property. The CBS inequality insures that $\|\mathbf{Ax}\|_2^2 = \sum_i |\mathbf{A}_{i*}\mathbf{x}|^2 \le \sum_i \|\mathbf{A}_{i*}\|_2^2 \|\mathbf{x}\|_2^2 = \|\mathbf{A}\|_F^2 \|\mathbf{x}\|_2^2$. That is,

$$\|\mathbf{Ax}\|_2 \le \|\mathbf{A}\|_F \|\mathbf{x}\|_2, \qquad (5.2.2)$$

and we express this by saying that the Frobenius matrix norm $\|\star\|_F$ and the euclidean vector norm $\|\star\|_2$ are ***compatible***. The compatibility condition (5.2.2) implies that for all conformable matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\begin{aligned}\|\mathbf{AB}\|_F^2 &= \sum_j \|[\mathbf{AB}]_{*j}\|_2^2 = \sum_j \|\mathbf{AB}_{*j}\|_2^2 \le \sum_j \|\mathbf{A}\|_F^2 \|\mathbf{B}_{*j}\|_2^2 \\ &= \|\mathbf{A}\|_F^2 \sum_j \|\mathbf{B}_{*j}\|_2^2 = \|\mathbf{A}\|_F^2 \|\mathbf{B}\|_F^2 \implies \|\mathbf{AB}\|_F \le \|\mathbf{A}\|_F \|\mathbf{B}\|_F.\end{aligned}$$

This suggests that the submultiplicative property $\|\mathbf{AB}\| \le \|\mathbf{A}\|\,\|\mathbf{B}\|$ should be added to (5.1.9) to define a general matrix norm.

General Matrix Norms

A ***matrix norm*** is a function $\|\star\|$ from the set of all complex matrices (of all finite orders) into $\Re$ that satisfies the following properties.

$$
\begin{aligned}
&\|\mathbf{A}\| \geq 0 \quad \text{and} \quad \|\mathbf{A}\| = 0 \Longleftrightarrow \mathbf{A} = \mathbf{0}.\\
&\|\alpha\mathbf{A}\| = |\alpha|\ \|\mathbf{A}\| \quad \text{for all scalars } \alpha.\\
&\|\mathbf{A}+\mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\| \quad \text{for matrices of the same size.}\\
&\|\mathbf{AB}\| \leq \|\mathbf{A}\|\ \|\mathbf{B}\| \quad \text{for all conformable matrices.}
\end{aligned}
\tag{5.2.3}
$$

The Frobenius norm satisfies the above definition (it was built that way), but where do other useful matrix norms come from? In fact, every legitimate vector norm generates (or induces) a matrix norm as described below.

Induced Matrix Norms

A vector norm that is defined on $\mathcal{C}^p$ for $p = m, n$ ***induces*** a matrix norm on $\mathcal{C}^{m\times n}$ by setting

$$\|\mathbf{A}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{Ax}\| \quad \text{for } \mathbf{A} \in \mathcal{C}^{m\times n},\ \ \mathbf{x} \in \mathcal{C}^{n\times 1}. \tag{5.2.4}$$

The footnote on p. 276 explains why this maximum value must exist.

- It's apparent that an induced matrix norm is compatible with its underlying vector norm in the sense that

$$\|\mathbf{Ax}\| \leq \|\mathbf{A}\|\,\|\mathbf{x}\|. \tag{5.2.5}$$

- When $\mathbf{A}$ is nonsingular, $\displaystyle\min_{\|\mathbf{x}\|=1} \|\mathbf{Ax}\| = \frac{1}{\|\mathbf{A}^{-1}\|}$. (5.2.6)

Proof. Verifying that $\max_{\|\mathbf{x}\|=1} \|\mathbf{Ax}\|$ satisfies the first three conditions in (5.2.3) is straightforward, and (5.2.5) implies $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\,\|\mathbf{B}\|$ (see Exercise 5.2.5). Property (5.2.6) is developed in Exercise 5.2.7. ∎

In words, an induced norm $\|\mathbf{A}\|$ represents the maximum extent to which a vector on the unit sphere can be stretched by $\mathbf{A}$, and $1/\left\|\mathbf{A}^{-1}\right\|$ measures the extent to which a nonsingular matrix $\mathbf{A}$ can shrink vectors on the unit sphere. Figure 5.2.1 depicts this in $\Re^3$ for the induced matrix 2-norm.

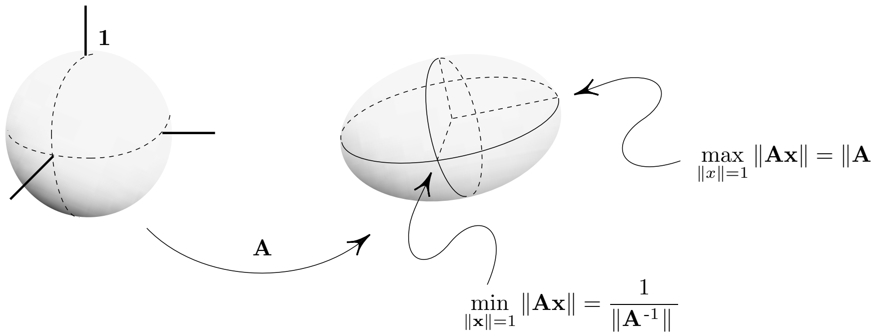

FIGURE 5.2.1. THE INDUCED MATRIX 2-NORM IN $\Re^3$.

Intuition might suggest that the euclidean vector norm should induce the Frobenius matrix norm (5.2.1), but something surprising happens instead.

Matrix 2-Norm

- The matrix norm induced by the euclidean vector norm is

$$\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2 = \sqrt{\lambda_{\max}}, \tag{5.2.7}$$

where $\lambda_{\max}$ is the largest number λ such that $\mathbf{A}^*\mathbf{A} - \lambda\mathbf{I}$ is singular.

- When $\mathbf{A}$ is nonsingular,

$$\left\|\mathbf{A}^{-1}\right\|_2 = \frac{1}{\displaystyle\min_{\|x\|_2=1} \|\mathbf{A}\mathbf{x}\|_2} = \frac{1}{\sqrt{\lambda_{\min}}}, \tag{5.2.8}$$

where $\lambda_{\min}$ is the smallest number λ such that $\mathbf{A}^*\mathbf{A} - \lambda\mathbf{I}$ is singular.

Note: If you are already familiar with eigenvalues, these say that $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of $\mathbf{A}^*\mathbf{A}$ (Example 7.5.1, p. 549), while $(\lambda_{\max})^{1/2} = \sigma_1$ and $(\lambda_{\min})^{1/2} = \sigma_n$ are the largest and smallest singular values of $\mathbf{A}$ (p. 414).

Proof. To prove (5.2.7), assume that $\mathbf{A}_{m\times n}$ is real (a proof for complex matrices is given in Example 7.5.1 on p. 549). The strategy is to evaluate $\|\mathbf{A}\|_2^2$ by solving the problem

$$\text{maximize } f(\mathbf{x}) = \|\mathbf{A}\mathbf{x}\|_2^2 = \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} \quad \text{subject to } g(\mathbf{x}) = \mathbf{x}^T\mathbf{x} = 1$$

using the method of Lagrange multipliers. Introduce a new variable λ (the Lagrange multiplier), and consider the function $h(\mathbf{x},\lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$. The points at which f is maximized are contained in the set of solutions to the equations $\partial h/\partial x_i = 0 \quad (i = 1,2,\ldots,n)$ along with $g(\mathbf{x}) = 1$. Differentiating h with respect to the x_i 's is essentially the same as described on p. 227, and the system generated by $\partial h/\partial x_i = 0$ $(i = 1,2,\ldots,n)$ is $(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. In other words, f is maximized at a vector $\mathbf{x}$ for which $(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ and $\|\mathbf{x}\|_2 = 1$. Consequently, λ must be a number such that $\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}$ is singular (because $\mathbf{x} \neq \mathbf{0}$). Since

$$\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x} = \lambda,$$

it follows that

$$\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\| = \max_{\|\mathbf{x}\|^2=1} \|\mathbf{A}\mathbf{x}\| = \left(\max_{\mathbf{x}^T\mathbf{x}=1} \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x}\right)^{1/2} = \sqrt{\lambda_{\max}},$$

where $\lambda_{\max}$ is the largest number λ for which $\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}$ is singular. A similar argument applied to (5.2.6) proves (5.2.8). Also, an independent development of (5.2.7) and (5.2.8) is contained in the discussion of singular values on p. 412. ■

Example 5.2.1

Problem: Determine the induced norm $\|\mathbf{A}\|_2$ as well as $\|\mathbf{A}^{-1}\|_2$ for the nonsingular matrix

$$\mathbf{A} = \frac{1}{\sqrt{3}}\begin{pmatrix} 3 & -1 \\ 0 & \sqrt{8} \end{pmatrix}.$$

Solution: Find the values of λ that make $\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}$ singular by applying Gaussian elimination to produce

$$\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 3-\lambda \\ 3-\lambda & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 3-\lambda \\ 0 & -1+(3-\lambda)^2 \end{pmatrix}.$$

This shows that $\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}$ is singular when $-1 + (3-\lambda)^2 = 0$ or, equivalently, when $\lambda = 2$ or $\lambda = 4$, so $\lambda_{\min} = 2$ and $\lambda_{\max} = 4$. Consequently, (5.2.7) and (5.2.8) say that

$$\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}} = 2 \quad \text{and} \quad \|\mathbf{A}^{-1}\|_2 = \frac{1}{\sqrt{\lambda_{\min}}} = \frac{1}{\sqrt{2}}.$$

Note: As mentioned earlier, the values of λ that make $\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}$ singular are called the *eigenvalues* of $\mathbf{A}^T\mathbf{A}$, and they are the focus of Chapter 7 where their determination is discussed in more detail. Using Gaussian elimination to determine the eigenvalues is not practical for larger matrices.

Some useful properties of the matrix 2-norm are stated below.

Properties of the 2-Norm

In addition to the properties shared by all induced norms, the 2-norm enjoys the following special properties.

- $\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2=1} \max_{\|\mathbf{y}\|_2=1} |\mathbf{y}^*\mathbf{A}\mathbf{x}|.$ (5.2.9)
- $\|\mathbf{A}\|_2 = \|\mathbf{A}^*\|_2.$ (5.2.10)
- $\|\mathbf{A}^*\mathbf{A}\|_2 = \|\mathbf{A}\|_2^2.$ (5.2.11)
- $\left\|\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}\right\|_2 = \max\left\{ \|\mathbf{A}\|_2, \|\mathbf{B}\|_2 \right\}.$ (5.2.12)
- $\|\mathbf{U}^*\mathbf{A}\mathbf{V}\|_2 = \|\mathbf{A}\|_2$ when $\mathbf{U}\mathbf{U}^* = \mathbf{I}$ and $\mathbf{V}^*\mathbf{V} = \mathbf{I}.$ (5.2.13)

You are asked to verify the validity of these properties in Exercise 5.2.6 on p. 285. Furthermore, some additional properties of the matrix 2-norm are developed in Exercise 5.6.9 and on pp. 414 and 417.

Now that we understand how the euclidean vector norm induces the matrix 2-norm, let's investigate the nature of the matrix norms that are induced by the vector 1-norm and the vector ∞-norm.

Matrix 1-Norm and Matrix ∞-Norm

The matrix norms induced by the vector 1-norm and ∞-norm are as follows.

- $$\|\mathbf{A}\|_1 = \max_{\|\mathbf{x}\|_1=1} \|\mathbf{A}\mathbf{x}\|_1 = \max_j \sum_i |a_{ij}| \\ = \text{the largest absolute column sum.} \tag{5.2.14}$$
- $$\|\mathbf{A}\|_\infty = \max_{\|\mathbf{x}\|_\infty=1} \|\mathbf{A}\mathbf{x}\|_\infty = \max_i \sum_j |a_{ij}| \\ = \text{the largest absolute row sum.} \tag{5.2.15}$$

Proof of (5.2.14). For all $\mathbf{x}$ with $\|\mathbf{x}\|_1 = 1$, the scalar triangle inequality yields

$$\|\mathbf{A}\mathbf{x}\|_1 = \sum_i |\mathbf{A}_{i*}\mathbf{x}| = \sum_i \Big| \sum_j a_{ij}x_j \Big| \le \sum_i \sum_j |a_{ij}|\,|x_j| = \sum_j \Big(|x_j| \sum_i |a_{ij}| \Big)$$
$$\le \Big(\sum_j |x_j| \Big) \Big(\max_j \sum_i |a_{ij}| \Big) = \max_j \sum_i |a_{ij}|.$$

Equality can be attained because if $\mathbf{A}_{*k}$ is the column with largest absolute sum, set $\mathbf{x} = \mathbf{e}_k$, and note that $\|\mathbf{e}_k\|_1 = 1$ and $\|\mathbf{A}\mathbf{e}_k\|_1 = \|\mathbf{A}_{*k}\|_1 = \max_j \sum_i |a_{ij}|$.

Proof of (5.2.15). For all $\mathbf{x}$ with $\|\mathbf{x}\|_\infty = 1$,

$$\|\mathbf{A}\mathbf{x}\|_\infty = \max_i \left| \sum_j a_{ij}x_j \right| \le \max_i \sum_j |a_{ij}| \; |x_j| \le \max_i \sum_j |a_{ij}| \,.$$

Equality can be attained because if $\mathbf{A}_{k*}$ is the row with largest absolute sum, and if $\mathbf{x}$ is the vector such that

$$x_j = \begin{cases} 1 & \text{if } a_{kj} \ge 0, \\ -1 & \text{if } a_{kj} < 0, \end{cases} \quad \text{then} \quad \begin{cases} |\mathbf{A}_{i*}\mathbf{x}| = |\sum_j a_{ij}x_j| \le \sum_j |a_{ij}| \text{ for all } i, \\ |\mathbf{A}_{k*}\mathbf{x}| = \sum_j |a_{kj}| = \max_i \sum_j |a_{ij}| \,, \end{cases}$$

so $\|\mathbf{x}\|_\infty = 1$, and $\|\mathbf{A}\mathbf{x}\|_\infty = \max_i |\mathbf{A}_{i*}\mathbf{x}| = \max_i \sum_j |a_{ij}|$. ■

Example 5.2.2

Problem: Determine the induced matrix norms $\|\mathbf{A}\|_1$ and $\|\mathbf{A}\|_\infty$ for

$$\mathbf{A} = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 & -1 \\ 0 & \sqrt{8} \end{pmatrix},$$

and compare the results with $\|\mathbf{A}\|_2$ (from Example 5.2.1) and $\|\mathbf{A}\|_F$.

Solution: Equation (5.2.14) says that $\|\mathbf{A}\|_1$ is the largest absolute column sum in $\mathbf{A}$, and (5.2.15) says that $\|\mathbf{A}\|_\infty$ is the largest absolute row sum, so

$$\|\mathbf{A}\|_1 = 1/\sqrt{3} + \sqrt{8}/\sqrt{3} \approx 2.21 \quad \text{and} \quad \|\mathbf{A}\|_\infty = 4/\sqrt{3} \approx 2.31.$$

Since $\|\mathbf{A}\|_2 = 2$ (Example 5.2.1) and $\|\mathbf{A}\|_F = \sqrt{trace\,(\mathbf{A}^T\mathbf{A})} = \sqrt{6} \approx 2.45$, we see that while $\|\mathbf{A}\|_1$, $\|\mathbf{A}\|_2$, $\|\mathbf{A}\|_\infty$, and $\|\mathbf{A}\|_F$ are not equal, they are all in the same ballpark. This is true for all $n \times n$ matrices because it can be shown that $\|\mathbf{A}\|_i \le \alpha \|\mathbf{A}\|_j$, where α is the (i, j)-entry in the following matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ \infty \\ F \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \infty & F \end{array} \\ \begin{pmatrix} * & \sqrt{n} & n & \sqrt{n} \\ \sqrt{n} & * & \sqrt{n} & 1 \\ n & \sqrt{n} & * & \sqrt{n} \\ \sqrt{n} & \sqrt{n} & \sqrt{n} & * \end{pmatrix} \end{array}$$

(see Exercise 5.1.8 and Exercise 5.12.3 on p. 425). Since it's often the case that only the order of magnitude of $\|\mathbf{A}\|$ is needed and not the exact value (e.g., recall the rule of thumb in Example 3.8.2 on p. 129), and since $\|\mathbf{A}\|_2$ is difficult to compute in comparison with $\|\mathbf{A}\|_1$, $\|\mathbf{A}\|_\infty$, and $\|\mathbf{A}\|_F$, you can see why any of these three might be preferred over $\|\mathbf{A}\|_2$ in spite of the fact that $\|\mathbf{A}\|_2$ is more "natural" by virtue of being induced by the euclidean vector norm.

Exercises for section 5.2

5.2.1. Evaluate the Frobenius matrix norm for each matrix below.

$$\mathbf{A}=\begin{pmatrix}1&-2\\-1&2\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\quad \mathbf{C}=\begin{pmatrix}4&-2&4\\-2&1&-2\\4&-2&4\end{pmatrix}.$$

5.2.2. Evaluate the induced 1-, 2-, and ∞-matrix norm for each of the three matrices given in Exercise 5.2.1.

5.2.3. (a) Explain why $\|\mathbf{I}\|=1$ for every induced matrix norm (5.2.4).
(b) What is $\|\mathbf{I}_{n\times n}\|_F$?

5.2.4. Explain why $\|\mathbf{A}\|_F=\|\mathbf{A}^*\|_F$ for Frobenius matrix norm (5.2.1).

5.2.5. For matrices $\mathbf{A}$ and $\mathbf{B}$ and for vectors $\mathbf{x}$, establish the following compatibility properties between a vector norm defined on every $\mathcal{C}^p$ and the associated induced matrix norm.
(a) Show that $\|\mathbf{Ax}\|\le\|\mathbf{A}\|\,\|\mathbf{x}\|$.
(b) Show that $\|\mathbf{AB}\|\le\|\mathbf{A}\|\,\|\mathbf{B}\|$.
(c) Explain why $\|\mathbf{A}\|=\max_{\|\mathbf{x}\|\le1}\|\mathbf{Ax}\|$.

5.2.6. Establish the following properties of the matrix 2-norm.
(a) $\|\mathbf{A}\|_2=\max\limits_{\substack{\|\mathbf{x}\|_2=1\\\|\mathbf{y}\|_2=1}}|\mathbf{y}^*\mathbf{Ax}|$,
(b) $\|\mathbf{A}\|_2=\|\mathbf{A}^*\|_2$,
(c) $\|\mathbf{A}^*\mathbf{A}\|_2=\|\mathbf{A}\|_2^2$,
(d) $\left\|\begin{pmatrix}\mathbf{A}&\mathbf{0}\\\mathbf{0}&\mathbf{B}\end{pmatrix}\right\|_2=\max\left\{\|\mathbf{A}\|_2,\|\mathbf{B}\|_2\right\}$ (take $\mathbf{A}$, $\mathbf{B}$ to be real),
(e) $\|\mathbf{U}^*\mathbf{AV}\|_2=\|\mathbf{A}\|_2$ when $\mathbf{UU}^*=\mathbf{I}$ and $\mathbf{V}^*\mathbf{V}=\mathbf{I}$.

5.2.7. Using the induced matrix norm (5.2.4), prove that if $\mathbf{A}$ is nonsingular, then

$$\|\mathbf{A}\|=\frac{1}{\min\limits_{\|\mathbf{x}\|=1}\left\|\mathbf{A}^{-1}\mathbf{x}\right\|}\quad\text{or, equivalently,}\quad\left\|\mathbf{A}^{-1}\right\|=\frac{1}{\min\limits_{\|\mathbf{x}\|=1}\|\mathbf{Ax}\|}.$$

5.2.8. For $\mathbf{A}\in\mathcal{C}^{n\times n}$ and a parameter $z\in\mathcal{C}$, the matrix $\mathbf{R}(z)=(z\mathbf{I}-\mathbf{A})^{-1}$ is called the ***resolvent of*** $\mathbf{A}$. Prove that if $|z|>\|\mathbf{A}\|$ for any induced matrix norm, then

$$\|\mathbf{R}(z)\|\le\frac{1}{|z|-\|\mathbf{A}\|}.$$

Similarly, $\sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p/q} \le \|\mathbf{y}\|_p \, \|\mathbf{x}+\mathbf{y}\|_p^{p-1}$, and therefore

$$\|\mathbf{x}+\mathbf{y}\|_p^p \le \left(\|\mathbf{x}\|_p + \|\mathbf{y}\|_p\right) \|\mathbf{x}+\mathbf{y}\|_p^{p-1} \quad \Longrightarrow \quad \|\mathbf{x}+\mathbf{y}\|_p \le \|\mathbf{x}\|_p + \|\mathbf{y}\|_p \, .$$

Solutions for exercises in section 5. 2

5.2.1. $\|\mathbf{A}\|_F = \left[\sum_{i,j} |a_{ij}|^2\right]^{1/2} = \left[\mathit{trace}\,(\mathbf{A}^*\mathbf{A})\right]^{1/2} = \sqrt{10}$,

$\|\mathbf{B}\|_F = \sqrt{3}$, and $\|\mathbf{C}\|_F = \sqrt{9}$.

5.2.2. (a) $\|\mathbf{A}\|_1$ = max absolute column sum = 4, and $\|\mathbf{A}\|_\infty$ = max absolute row sum = 3. $\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the largest value of λ for which $\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}$ is singular. Determine these λ's by row reduction.

$$\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2--\lambda & -4 \\ -4 & 8-\lambda \end{pmatrix} \longrightarrow \begin{pmatrix} -4 & 8-\lambda \\ 2-\lambda & -4 \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} -4 & 8-\lambda \\ 0 & -4 + \frac{2-\lambda}{4}(8-\lambda) \end{pmatrix}$$

This matrix is singular if and only if the second pivot is zero, so we must have $(2-\lambda)(8-\lambda) - 16 = 0 \implies \lambda^2 - 10\lambda = 0 \implies \lambda = 0, \lambda = 10$, and therefore $\|\mathbf{A}\|_2 = \sqrt{10}$.

(b) Use the same technique to get $\|\mathbf{B}\|_1 = \|\mathbf{B}\|_2 = \|\mathbf{B}\|_\infty = 1$, and

(c) $\|\mathbf{C}\|_1 = \|\mathbf{C}\|_\infty = 10$ and $\|\mathbf{C}\|_2 = 9$.

5.2.3. (a) $\|\mathbf{I}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{I}\mathbf{x}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{x}\| = 1$.

(b) $\|\mathbf{I}_{n\times n}\|_F = \left[\mathit{trace}\left(\mathbf{I}^T\mathbf{I}\right)\right]^{1/2} = \sqrt{n}$.

5.2.4. Use the fact that $\mathit{trace}\,(\mathbf{AB}) = \mathit{trace}\,(\mathbf{BA})$ (recall Example 3.6.5) to write

$$\|\mathbf{A}\|_F^2 = \mathit{trace}\,(\mathbf{A}^*\mathbf{A}) = \mathit{trace}\,(\mathbf{A}\mathbf{A}^*) = \|\mathbf{A}^*\|_F^2 \, .$$

5.2.5. (a) For $\mathbf{x} = \mathbf{0}$, the statement is trivial. For $\mathbf{x} \ne \mathbf{0}$, we have $\|(\mathbf{x}/\|\mathbf{x}\|)\| = 1$, so for any particular $\mathbf{x}_0 \ne \mathbf{0}$,

$$\|\mathbf{A}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\| = \max_{\mathbf{x}\ne\mathbf{0}} \left\| \mathbf{A}\frac{\mathbf{x}}{\|\mathbf{x}\|} \right\| \ge \frac{\|\mathbf{A}\mathbf{x}_0\|}{\|\mathbf{x}_0\|} \quad \Longrightarrow \quad \|\mathbf{A}\mathbf{x}_0\| \le \|\mathbf{A}\| \, \|\mathbf{x}_0\| \, .$$

(b) Let $\mathbf{x}_0$ be a vector such that $\|\mathbf{x}_0\| = 1$ and

$$\|\mathbf{AB}\mathbf{x}_0\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{AB}\mathbf{x}\| = \|\mathbf{AB}\| \, .$$

Make use of the result of part (a) to write

$$\|\mathbf{AB}\| = \|\mathbf{AB}\mathbf{x}_0\| \le \|\mathbf{A}\| \, \|\mathbf{B}\mathbf{x}_0\| \le \|\mathbf{A}\| \, \|\mathbf{B}\| \, \|\mathbf{x}_0\| = \|\mathbf{A}\| \, \|\mathbf{B}\| \, .$$

(c) $\|\mathbf{A}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\| \le \max_{\|\mathbf{x}\|\le 1} \|\mathbf{A}\mathbf{x}\|$ because $\{\mathbf{x} \mid \|\mathbf{x}\| = 1\} \subset \{\mathbf{x} \mid \|\mathbf{x}\| \le 1\}$. If there would exist a vector $\mathbf{x}_0$ such that $\|\mathbf{x}_0\| < 1$ and $\|\mathbf{A}\| < \|\mathbf{A}\mathbf{x}_0\|$, then part (a) would insure that $\|\mathbf{A}\| < \|\mathbf{A}\mathbf{x}_0\| \le \|\mathbf{A}\| \|\mathbf{x}_0\| < \|\mathbf{A}\|$, which is impossible.

5.2.6. (a) Applying the CBS inequality yields

$$|\mathbf{y}^*\mathbf{A}\mathbf{x}| \le \|\mathbf{y}\|_2 \|\mathbf{A}\mathbf{x}\|_2 \implies \max_{\substack{\|\mathbf{x}\|_2=1\\ \|\mathbf{y}\|_2=1}} |\mathbf{y}^*\mathbf{A}\mathbf{x}| \le \max_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{A}\|_2 .$$

Now show that equality is actually attained for some pair $\mathbf{x}$ and $\mathbf{y}$ on the unit 2-sphere. To do so, notice that if $\mathbf{x}_0$ is a vector of unit length such that

$$\|\mathbf{A}\mathbf{x}_0\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{A}\|_2 , \quad \text{and if} \quad \mathbf{y}_0 = \frac{\mathbf{A}\mathbf{x}_0}{\|\mathbf{A}\mathbf{x}_0\|_2} = \frac{\mathbf{A}\mathbf{x}_0}{\|\mathbf{A}\|_2},$$

then

$$\mathbf{y}_0^*\mathbf{A}\mathbf{x}_0 = \frac{\mathbf{x}_0^*\mathbf{A}^*\mathbf{A}\mathbf{x}_0}{\|\mathbf{A}\|_2} = \frac{\|\mathbf{A}\mathbf{x}_0\|_2^2}{\|\mathbf{A}\|_2} = \frac{\|\mathbf{A}\|_2^2}{\|\mathbf{A}\|_2} = \|\mathbf{A}\|_2 .$$

(b) This follows directly from the result of part (a) because

$$\|\mathbf{A}\|_2 = \max_{\substack{\|\mathbf{x}\|_2=1\\ \|\mathbf{y}\|_2=1}} |\mathbf{y}^*\mathbf{A}\mathbf{x}| = \max_{\substack{\|\mathbf{x}\|_2=1\\ \|\mathbf{y}\|_2=1}} |(\mathbf{y}^*\mathbf{A}\mathbf{x})^*| = \max_{\substack{\|\mathbf{x}\|_2=1\\ \|\mathbf{y}\|_2=1}} |\mathbf{x}^*\mathbf{A}^*\mathbf{y}| = \|\mathbf{A}^*\|_2 .$$

(c) Use part (a) with the CBS inequality to write

$$\|\mathbf{A}^*\mathbf{A}\|_2 = \max_{\substack{\|\mathbf{x}\|_2=1\\ \|\mathbf{y}\|_2=1}} |\mathbf{y}^*\mathbf{A}^*\mathbf{A}\mathbf{x}| \le \max_{\substack{\|\mathbf{x}\|_2=1\\ \|\mathbf{y}\|_2=1}} \|\mathbf{A}\mathbf{y}\|_2 \|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{A}\|_2^2 .$$

To see that equality is attained, let $\mathbf{x} = \mathbf{y} = \mathbf{x}_0$, where $\mathbf{x}_0$ is a vector of unit length such that $\|\mathbf{A}\mathbf{x}_0\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{A}\|_2$, and observe

$$|\mathbf{x}_0^*\mathbf{A}^*\mathbf{A}\mathbf{x}_0| = \mathbf{x}_0^*\mathbf{A}^*\mathbf{A}\mathbf{x}_0 = \|\mathbf{A}\mathbf{x}_0\|_2^2 = \|\mathbf{A}\|_2^2 .$$

(d) Let $\mathbf{D} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}$. We know from (5.2.7) that $\|\mathbf{D}\|_2^2$ is the largest value λ such that $\mathbf{D}^T\mathbf{D} - \lambda\mathbf{I}$ is singular. But $\mathbf{D}^T\mathbf{D} - \lambda\mathbf{I}$ is singular if and only if $\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}$ or $\mathbf{B}^T\mathbf{B} - \lambda\mathbf{I}$ is singular, so $\lambda_{\max}(\mathbf{D}) = \max\{\lambda_{\max}(\mathbf{A}), \lambda_{\max}(\mathbf{B})\}$.

(e) If $\mathbf{U}\mathbf{U}^* = \mathbf{I}$, then $\|\mathbf{U}^*\mathbf{A}\mathbf{x}\|_2^2 = \mathbf{x}^*\mathbf{A}^*\mathbf{U}\mathbf{U}^*\mathbf{A}\mathbf{x} = \mathbf{x}^*\mathbf{A}^*\mathbf{A}\mathbf{x} = \|\mathbf{A}\mathbf{x}\|_2^2$, so $\|\mathbf{U}^*\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{U}^*\mathbf{A}\mathbf{x}\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{A}\|_2$. Now, if $\mathbf{V}^*\mathbf{V} = \mathbf{I}$, use what was just established with part (b) to write

$$\|\mathbf{A}\mathbf{V}\|_2 = \|(\mathbf{A}\mathbf{V})^*\|_2 = \|\mathbf{V}^*\mathbf{A}^*\|_2 = \|\mathbf{A}^*\|_2 = \|\mathbf{A}\|_2 \implies \|\mathbf{U}^*\mathbf{A}\mathbf{V}\|_2 = \|\mathbf{A}\|_2.$$

5.2.7. Proceed as follows.

$$\frac{1}{\min\limits_{\|\mathbf{x}\|=1} \left\|\mathbf{A}^{-1}\mathbf{x}\right\|} = \max_{\|\mathbf{x}\|=1} \left\{ \frac{1}{\|\mathbf{A}^{-1}\mathbf{x}\|} \right\} = \max_{\mathbf{y}\neq\mathbf{0}} \left\{ \frac{1}{\left\|\mathbf{A}^{-1}\frac{(\mathbf{A}\mathbf{y})}{\|\mathbf{A}\mathbf{y}\|}\right\|} \right\}$$

$$= \max_{\mathbf{y}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{y}\|}{\|\mathbf{A}^{-1}(\mathbf{A}\mathbf{y})\|} = \max_{\mathbf{y}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{y}\|}{\|\mathbf{y}\|} = \max_{\mathbf{y}\neq\mathbf{0}} \left\|\mathbf{A}\left(\frac{\mathbf{y}}{\|\mathbf{y}\|}\right)\right\|$$

$$= \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\| = \|\mathbf{A}\|$$

5.2.8. Use (5.2.6) on p. 280 to write $\|(z\mathbf{I}-\mathbf{A})^{-1}\| = (1/\min_{\|\mathbf{x}\|=1} \|(z\mathbf{I}-\mathbf{A})\mathbf{x}\|)$, and let $\mathbf{w}$ be a vector for which $\|\mathbf{w}\| = 1$ and $\|(z\mathbf{I}-\mathbf{A})\mathbf{w}\| = \min_{\|\mathbf{x}\|=1} \|(z\mathbf{I}-\mathbf{A})\mathbf{x}\|$. Use $\|\mathbf{A}\mathbf{w}\| \leq \|\mathbf{A}\| < |z|$ together with the "backward triangle inequality" from Example 5.1.1 (p. 273) to write

$$\|(z\mathbf{I}-\mathbf{A})\mathbf{w}\| = \|z\mathbf{w}-\mathbf{A}\mathbf{w}\| \geq \big|\|z\mathbf{w}\| - \|\mathbf{A}\mathbf{w}\|\big| = \big||z| - \|\mathbf{A}\mathbf{w}\|\big|$$
$$= |z| - \|\mathbf{A}\mathbf{w}\| \geq |z| - \|\mathbf{A}\|.$$

Consequently, $\min_{\|\mathbf{x}\|=1} \|(z\mathbf{I}-\mathbf{A})\mathbf{x}\| = \|(z\mathbf{I}-\mathbf{A})\mathbf{w}\| \geq |z| - \|\mathbf{A}\|$ implies that

$$\|(z\mathbf{I}-\mathbf{A})^{-1}\| = \frac{1}{\min\limits_{\|\mathbf{x}\|=1} \|(z\mathbf{I}-\mathbf{A})\mathbf{x}\|} \leq \frac{1}{|z| - \|\mathbf{A}\|}.$$

Solutions for exercises in section 5. 3

5.3.1. Only (c) is an inner product. The expressions in (a) and (b) each fail the first condition of the definition (5.3.1), and (d) fails the second.

5.3.2. (a) $\langle\mathbf{x}|\mathbf{y}\rangle = 0 \quad \forall\ \mathbf{x}\in\mathcal{V} \implies \langle\mathbf{y}|\mathbf{y}\rangle = 0 \implies \mathbf{y}=\mathbf{0}$.

(b) $\langle\alpha\mathbf{x}|\mathbf{y}\rangle = \overline{\langle\mathbf{y}|\alpha\mathbf{x}\rangle} = \overline{\alpha\,\langle\mathbf{y}|\mathbf{x}\rangle} = \overline{\alpha}\overline{\langle\mathbf{y}|\mathbf{x}\rangle} = \overline{\alpha}\,\langle\mathbf{x}|\mathbf{y}\rangle$

(c) $\langle\mathbf{x}+\mathbf{y}|\mathbf{z}\rangle = \overline{\langle\mathbf{z}|\mathbf{x}+\mathbf{y}\rangle} = \overline{\langle\mathbf{z}|\mathbf{x}\rangle + \langle\mathbf{z}|\mathbf{y}\rangle} = \overline{\langle\mathbf{z}|\mathbf{x}\rangle} + \overline{\langle\mathbf{z}|\mathbf{y}\rangle} = \langle\mathbf{x}|\mathbf{z}\rangle + \langle\mathbf{y}|\mathbf{z}\rangle$

5.3.3. The first property in (5.2.3) holds because $\langle\mathbf{x}|\mathbf{x}\rangle \geq 0$ for all $\mathbf{x}\in\mathcal{V}$ implies $\|\mathbf{x}\| = \sqrt{\langle\mathbf{x}|\mathbf{x}\rangle} \geq 0$, and $\|\mathbf{x}\| = 0 \iff \langle\mathbf{x}|\mathbf{x}\rangle = 0 \iff \mathbf{x}=\mathbf{0}$. The second property in (5.2.3) holds because

$$\|\alpha\mathbf{x}\|^2 = \langle\alpha\mathbf{x}|\alpha\mathbf{x}\rangle = \alpha\,\langle\alpha\mathbf{x}|\mathbf{x}\rangle = \alpha\overline{\langle\mathbf{x}|\alpha\mathbf{x}\rangle} = \alpha\overline{\alpha}\overline{\langle\mathbf{x}|\mathbf{x}\rangle} = |\alpha|^2\,\langle\mathbf{x}|\mathbf{x}\rangle = |\alpha|^2\,\|\mathbf{x}\|^2.$$

5.3.4. $0 \leq \|\mathbf{x}-\mathbf{y}\|^2 = \langle\mathbf{x}-\mathbf{y}|\mathbf{x}-\mathbf{y}\rangle = \langle\mathbf{x}|\mathbf{x}\rangle - 2\,\langle\mathbf{x}|\mathbf{y}\rangle + \langle\mathbf{y}|\mathbf{y}\rangle = \|\mathbf{x}\|^2 - 2\,\langle\mathbf{x}|\mathbf{y}\rangle + \|\mathbf{y}\|^2$

5.3.5. (a) Use the CBS inequality with the Frobenius matrix norm and the standard inner product as illustrated in Example 5.3.3, and set $\mathbf{A}=\mathbf{I}$.

(b) Proceed as in part (a), but this time set $\mathbf{A}=\mathbf{B}^T$ (recall from Example 3.6.5 that $trace\,(\mathbf{B}^T\mathbf{B}) = trace\,(\mathbf{B}\mathbf{B}^T)$).