6.2 ADDITIONAL PROPERTIES OF DETERMINANTS

The purpose of this section is to present some additional properties of determinants that will be helpful in later developments.

Block Determinants

If $\mathbf{A}$ and $\mathbf{D}$ are square matrices, then

$$\det\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix} = \begin{cases}\det(\mathbf{A})\det\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right) & \text{when } \mathbf{A}^{-1} \text{ exists,}\\ \det(\mathbf{D})\det\left(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right) & \text{when } \mathbf{D}^{-1} \text{ exists.}\end{cases} \tag{6.2.1}$$

The matrices $\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ are called the ***Schur complements*** of $\mathbf{A}$ and $\mathbf{D}$, respectively—see Exercise 3.7.11 on p. 123.

Proof. If $\mathbf{A}^{-1}$ exists, then $\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix} = \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{C}\mathbf{A}^{-1} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{0} & \mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\end{pmatrix}$, and the product rules (p. 467) produce the first formula in (6.2.1). The second formula follows by using a similar trick. ∎

Since the determinant of a product is equal to the product of the determinants, it's only natural to inquire if a similar result holds for sums. In other words, is $\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A})+\det(\mathbf{B})$? *Almost never*! Try a couple of examples to convince yourself. Nevertheless, there are still some statements that can be made regarding the determinant of certain types of sums. In a loose sense, the result of Exercise 6.1.14 was a statement concerning determinants and sums, but the following result is a little more satisfying.

Rank-One Updates

If $\mathbf{A}_{n\times n}$ is nonsingular, and if $\mathbf{c}$ and $\mathbf{d}$ are $n\times 1$ columns, then

- $\det\left(\mathbf{I}+\mathbf{c}\mathbf{d}^T\right) = 1+\mathbf{d}^T\mathbf{c},$ (6.2.2)
- $\det\left(\mathbf{A}+\mathbf{c}\mathbf{d}^T\right) = \det(\mathbf{A})\left(1+\mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}\right).$ (6.2.3)

Exercise 6.2.7 presents a generalized version of these formulas.

Proof. The proof of (6.2.2) follows by applying the product rules (p. 467) to

$$\begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{d}^T & 1\end{pmatrix}\begin{pmatrix}\mathbf{I}+\mathbf{c}\mathbf{d}^T & \mathbf{c}\\ \mathbf{0} & 1\end{pmatrix}\begin{pmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{d}^T & 1\end{pmatrix} = \begin{pmatrix}\mathbf{I} & \mathbf{c}\\ \mathbf{0} & 1+\mathbf{d}^T\mathbf{c}\end{pmatrix}.$$

To prove (6.2.3), write $\mathbf{A}+\mathbf{c}\mathbf{d}^T = \mathbf{A}\left(\mathbf{I}+\mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T\right)$, and apply the product rule (6.1.15) along with (6.2.2). ∎

Example 6.2.1

Problem: For $\mathbf{A} = \begin{pmatrix} 1+\lambda_1 & 1 & \cdots & 1 \\ 1 & 1+\lambda_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+\lambda_n \end{pmatrix}$, $\lambda_i \neq 0$, find $\det(\mathbf{A})$.

Solution: Express $\mathbf{A}$ as a rank-one updated matrix $\mathbf{A} = \mathbf{D} + \mathbf{e}\mathbf{e}^T$, where $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mathbf{e}^T = (1 \quad 1 \quad \cdots \quad 1)$. Apply (6.2.3) to produce

$$\det(\mathbf{D} + \mathbf{e}\mathbf{e}^T) = \det(\mathbf{D})\left(1 + \mathbf{e}^T\mathbf{D}^{-1}\mathbf{e}\right) = \left(\prod_{i=1}^{n} \lambda_i\right)\left(1 + \sum_{i=1}^{n} \frac{1}{\lambda_i}\right).$$

The classical result known as Cramer's rule[65] is a corollary of the rank-one update formula (6.2.3).

Cramer's Rule

In a nonsingular system $\mathbf{A}_{n\times n}\mathbf{x} = \mathbf{b}$, the i^{th} unknown is

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})},$$

where $\mathbf{A}_i = \left[\mathbf{A}_{*1} \,\middle|\, \cdots \,\middle|\, \mathbf{A}_{*i-1} \,\middle|\, \mathbf{b} \,\middle|\, \mathbf{A}_{*i+1} \,\middle|\, \cdots \,\middle|\, \mathbf{A}_{*n}\right]$. That is, $\mathbf{A}_i$ is identical to $\mathbf{A}$ except that column $\mathbf{A}_{*i}$ has been replaced by $\mathbf{b}$.

Proof. Since $\mathbf{A}_i = \mathbf{A} + (\mathbf{b} - \mathbf{A}_{*i})\,\mathbf{e}_i^T$, where $\mathbf{e}_i$ is the i^{th} unit vector, (6.2.3) may be applied to yield

$$\begin{aligned} \det(\mathbf{A}_i) &= \det(\mathbf{A})\Big(1 + \mathbf{e}_i^T\mathbf{A}^{-1}(\mathbf{b} - \mathbf{A}_{*i})\Big) = \det(\mathbf{A})\Big(1 + \mathbf{e}_i^T(\mathbf{x} - \mathbf{e}_i)\Big) \\ &= \det(\mathbf{A})(1 + x_i - 1) = \det(\mathbf{A})\,x_i. \end{aligned}$$

Thus $x_i = \det(\mathbf{A}_i)/\det(\mathbf{A})$ because $\mathbf{A}$ being nonsingular insures $\det(\mathbf{A}) \neq 0$ by (6.1.13). ■

---

[65] Gabriel Cramer (1704–1752) was a mathematician from Geneva, Switzerland. As mentioned in §6.1, Cramer's rule was apparently known to others long before Cramer rediscovered and published it in 1750. Nevertheless, Cramer's recognition is not undeserved because his work was responsible for a revived interest in determinants and systems of linear equations. After Cramer's publication, Cramer's rule met with instant success, and it quickly found its way into the textbooks and classrooms of Europe. It is reported that there was a time when students passed or failed the exams in the schools of public service in France according to their understanding of Cramer's rule.

Example 6.2.2

Problem: Determine the value of t for which $x_3(t)$ is minimized in

$$\begin{pmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1/t \\ 1/t^2 \end{pmatrix}.$$

Solution: Only one component of the solution is required, so it's wasted effort to solve the entire system. Use Cramer's rule to obtain

$$x_3(t) = \frac{\begin{vmatrix} t & 0 & 1 \\ 0 & t & 1/t \\ 1 & t^2 & 1/t^2 \end{vmatrix}}{\begin{vmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{vmatrix}} = \frac{1-t-t^2}{-1} = t^2+t-1, \quad \text{and set} \quad \frac{d\,x_3(t)}{dt} = 0$$

to conclude that $x_3(t)$ is minimized at $t = -1/2$.

Recall that minor determinants of $\mathbf{A}$ are simply determinants of submatrices of $\mathbf{A}$. We are now in a position to see that in an $n \times n$ matrix the $n-1 \times n-1$ minor determinants have a special significance.

Cofactors

The ***cofactor*** of $\mathbf{A}_{n\times n}$ associated with the (i, j)-position is defined as

$$\mathring{A}_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the $n-1 \times n-1$ minor obtained by deleting the i^{th} row and j^{th} column of $\mathbf{A}$. The matrix of cofactors is denoted by $\mathring{\mathbf{A}}$.

Example 6.2.3

Problem: For $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 6 \\ -3 & 9 & 1 \end{pmatrix}$, determine the cofactors $\mathring{A}_{21}$ and $\mathring{A}_{13}$.

Solution:

$\mathring{A}_{21} = (-1)^{2+1} M_{21} = (-1)(-19) = 19$ and $\mathring{A}_{13} = (-1)^{1+3} M_{13} = (+1)(18) = 18$.

The entire matrix of cofactors is $\mathring{\mathbf{A}} = \begin{pmatrix} -54 & -20 & 18 \\ 19 & 7 & -6 \\ -6 & -2 & 2 \end{pmatrix}$.

The cofactors of a square matrix $\mathbf{A}$ appear naturally in the expansion of $\det(\mathbf{A})$. For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{aligned} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\qquad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) \\ &\qquad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}\mathring{A}_{11} + a_{12}\mathring{A}_{12} + a_{13}\mathring{A}_{13}. \end{aligned} \tag{6.2.4}$$

Because this expansion is in terms of the entries of the first row and the corresponding cofactors, (6.2.4) is called *the cofactor expansion of* $\det(\mathbf{A})$ *in terms of the first row.* It should be clear that there is nothing special about the first row of $\mathbf{A}$. That is, it's just as easy to write an expression similar to (6.2.4) in which entries from any other row or column appear. For example, the terms in (6.2.4) can be rearranged to produce

$$\begin{aligned} \det(\mathbf{A}) &= a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) + a_{32}(a_{13}a_{21} - a_{11}a_{23}) \\ &= a_{12}\mathring{A}_{12} + a_{22}\mathring{A}_{22} + a_{32}\mathring{A}_{32}. \end{aligned}$$

This is called *the cofactor expansion for* $\det(\mathbf{A})$ *in terms of the second column.* The 3×3 case is typical, and exactly the same reasoning can be applied to a more general $n \times n$ matrix in order to obtain the following statements.

Cofactor Expansions

- $\det(\mathbf{A}) = a_{i1}\mathring{A}_{i1} + a_{i2}\mathring{A}_{i2} + \cdots + a_{in}\mathring{A}_{in}$ (about row i). (6.2.5)
- $\det(\mathbf{A}) = a_{1j}\mathring{A}_{1j} + a_{2j}\mathring{A}_{2j} + \cdots + a_{nj}\mathring{A}_{nj}$ (about column j). (6.2.6)

Example 6.2.4

Problem: Use cofactor expansions to evaluate $\det(\mathbf{A})$ for

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 7 & 1 & 6 & 5 \\ 3 & 7 & 2 & 0 \\ 0 & 3 & -1 & 4 \end{pmatrix}.$$

Solution: To minimize the effort, expand $\det(\mathbf{A})$ in terms of the row or column that contains a maximal number of zeros. For this example, the expansion in terms of the first row is most efficient because

$$\det(\mathbf{A}) = a_{11}\mathring{A}_{11} + a_{12}\mathring{A}_{12} + a_{13}\mathring{A}_{13} + a_{14}\mathring{A}_{14} = a_{14}\mathring{A}_{14} = (2)(-1)\begin{vmatrix} 7 & 1 & 6 \\ 3 & 7 & 2 \\ 0 & 3 & -1 \end{vmatrix}.$$

Now expand this remaining 3×3 determinant either in terms of the first column or the third row. Using the first column produces

$$\begin{vmatrix} 7 & 1 & 6 \\ 3 & 7 & 2 \\ 0 & 3 & -1 \end{vmatrix} = (7)(+1)\begin{vmatrix} 7 & 2 \\ 3 & -1 \end{vmatrix} + (3)(-1)\begin{vmatrix} 1 & 6 \\ 3 & -1 \end{vmatrix} = -91 + 57 = -34,$$

so $\det(\mathbf{A}) = (2)(-1)(-34) = 68$. You may wish to try an expansion using different rows or columns, and verify that the final result is the same.

In the previous example, we were able to take advantage of the fact that there were zeros in convenient positions. However, for a general matrix $\mathbf{A}_{n\times n}$ with no zero entries, it's not difficult to verify that successive application of cofactor expansions requires $n!\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!}\right)$ multiplications to evaluate $\det(\mathbf{A})$. Even for moderate values of n, this number is too large for the cofactor expansion to be practical for computational purposes. Nevertheless, cofactors can be useful for theoretical developments such as the following determinant formula for $\mathbf{A}^{-1}$.

Determinant Formula for $\mathbf{A}^{-1}$

The ***adjugate*** of $\mathbf{A}_{n\times n}$ is defined to be $\text{adj}(\mathbf{A}) = \mathring{\mathbf{A}}^T$, the transpose of the matrix of cofactors—some older texts call this the ***adjoint*** matrix. If $\mathbf{A}$ is nonsingular, then

$$\mathbf{A}^{-1} = \frac{\mathring{\mathbf{A}}^T}{\det(\mathbf{A})} = \frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})}. \tag{6.2.7}$$

Proof. $\left[\mathbf{A}^{-1}\right]_{ij}$ is the i^{th} component in the solution to $\mathbf{Ax} = \mathbf{e}_j$, where $\mathbf{e}_j$ is the j^{th} unit vector. By Cramer's rule, this is

$$\left[\mathbf{A}^{-1}\right]_{ij} = x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})},$$

where $\mathbf{A}_i$ is identical to $\mathbf{A}$ except that the i^{th} column has been replaced by $\mathbf{e}_j$, and the cofactor expansion in terms of the i^{th} column implies that

$$\det(\mathbf{A}_i) = \begin{vmatrix} a_{11} & \cdots & \overset{\overset{i^{th}}{\downarrow}}{0} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{j1} & \cdots & 1 & \cdots & a_{jn} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix} = \mathring{A}_{ji}. \quad \blacksquare$$

Example 6.2.5

Problem: Use determinants to compute $[\mathbf{A}^{-1}]_{12}$ and $[\mathbf{A}^{-1}]_{31}$ for the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 6 \\ -3 & 9 & 1 \end{pmatrix}.$$

Solution: The cofactors $\mathring{A}_{21}$ and $\mathring{A}_{13}$ were determined in Example 6.2.3 to be $\mathring{A}_{21} = 19$ and $\mathring{A}_{13} = 18$, and it's straightforward to compute $\det(\mathbf{A}) = 2$, so

$$[\mathbf{A}^{-1}]_{12} = \frac{\mathring{A}_{21}}{\det(\mathbf{A})} = \frac{19}{2} \quad \text{and} \quad [\mathbf{A}^{-1}]_{31} = \frac{\mathring{A}_{13}}{\det(\mathbf{A})} = \frac{18}{2} = 9.$$

Using the matrix of cofactors $\mathring{\mathbf{A}}$ computed in Example 6.2.3, we have that

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}(\mathbf{A})}{\det(\mathbf{A})} = \frac{\mathring{\mathbf{A}}^T}{\det(\mathbf{A})} = \frac{1}{2}\begin{pmatrix} -54 & 19 & -6 \\ -20 & 7 & -2 \\ 18 & -6 & 2 \end{pmatrix}.$$

Example 6.2.6

Problem: For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, determine a general formula for $\mathbf{A}^{-1}$.

Solution: $\operatorname{adj}(\mathbf{A}) = \mathring{\mathbf{A}}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and $\det(\mathbf{A}) = ad - bc$, so

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}(\mathbf{A})}{\det(\mathbf{A})} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example 6.2.7

Problem: Explain why the entries in $\mathbf{A}^{-1}$ vary continuously with the entries in $\mathbf{A}$ when $\mathbf{A}$ is nonsingular. This is in direct contrast with the lack of continuity exhibited by pseudoinverses (p. 423).

Solution: Recall from elementary calculus that the sum, the product, and the quotient of continuous functions are each continuous functions. In particular, the sum and the product of any set of numbers varies continuously as the numbers vary, so $\det(\mathbf{A})$ is a continuous function of the a_{ij}'s. Since each entry in $\operatorname{adj}(\mathbf{A})$ is a determinant, each quotient $[\mathbf{A}^{-1}]_{ij} = [\operatorname{adj}(\mathbf{A})]_{ij}/\det(\mathbf{A})$ must be a continuous function of the a_{ij}'s.

The Moral: The formula $\mathbf{A}^{-1} = \operatorname{adj}(\mathbf{A})/\det(\mathbf{A})$ is nearly worthless for actually computing the value of $\mathbf{A}^{-1}$, but, as this example demonstrates, the formula is nevertheless a useful mathematical tool. It's not uncommon for applied oriented students to fall into the trap of believing that the worth of a formula or an idea is tied to its utility for computing something. This example makes the point that things can have significant mathematical value without being computationally important. In fact, most of this chapter is in this category.

Example 6.2.8

Problem: Explain why the inner product of one row (or column) in $\mathbf{A}_{n\times n}$ with the cofactors of a different row (or column) in $\mathbf{A}$ must always be zero.

Solution: Let $\tilde{\mathbf{A}}$ be the result of replacing the j^{th} column in $\mathbf{A}$ by the k^{th} column of $\mathbf{A}$. Since $\tilde{\mathbf{A}}$ has two identical columns, $\det(\tilde{\mathbf{A}}) = 0$. Furthermore, the cofactor associated with the (i, j)-position in $\tilde{\mathbf{A}}$ is $\mathring{A}_{ij}$, the cofactor associated with the (i, j) in $\mathbf{A}$, so expansion of $\det(\tilde{\mathbf{A}})$ in terms of the j^{th} column yields

$$0 = \det(\tilde{\mathbf{A}}) = \begin{vmatrix} a_{11} & \cdots & \overset{j^{th}\ \downarrow}{a_{1k}} & \cdots & \overset{k^{th}\ \downarrow}{a_{1k}} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots & & \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots & & \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nk} & \cdots & a_{nn} \end{vmatrix} = \sum_{i=1}^{n} a_{ik}\mathring{A}_{ij}.$$

Thus the inner product of the k^{th} column of $\mathbf{A}_{n\times n}$ with the cofactors of the j^{th} column of $\mathbf{A}$ is zero. A similar result holds for rows. Combining these observations with (6.2.5) and (6.2.6) produces

$$\sum_{j=1}^{n} a_{kj}\mathring{A}_{ij} = \begin{cases} \det(\mathbf{A}) & \text{if } k = i, \\ 0 & \text{if } k \neq i, \end{cases} \quad \text{and} \quad \sum_{i=1}^{n} a_{ik}\mathring{A}_{ij} = \begin{cases} \det(\mathbf{A}) & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases}$$

which is equivalent to saying that $\mathbf{A}[\operatorname{adj}(\mathbf{A})] = [\operatorname{adj}(\mathbf{A})]\mathbf{A} = \det(\mathbf{A})\,\mathbf{I}$.

Example 6.2.9

Differential Equations and Determinants. A system of n homogeneous first-order linear differential equations

$$\frac{d\,x_i(t)}{dt} = a_{i1}(t)x_1(t) + a_{i2}(t)x_2(t) + \cdots + a_{in}(t)x_n(t), \quad i = 1, 2, \ldots, n$$

can be expressed in matrix notation by writing

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

or, equivalently, $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Let $\mathcal{S} = \{\mathbf{w}_1(t), \mathbf{w}_2(t), \ldots, \mathbf{w}_n(t)\}$ be a set of $n \times 1$ vectors that are solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and place these solutions as columns in a matrix $\mathbf{W}(t)_{n\times n} = [\mathbf{w}_1(t)\,|\,\mathbf{w}_2(t)\,|\cdots|\,\mathbf{w}_n(t)]$ so that $\mathbf{W}' = \mathbf{A}\mathbf{W}$.

Problem: Prove that if $w(t) = \det(\mathbf{W})$, (called the *Wronskian* (p. 474)), then

$$w(t) = w(\xi_0)\, e^{\int_{\xi_0}^{t} \operatorname{trace} \mathbf{A}(\xi)\, d\xi}, \quad \text{where } \xi_0 \text{ is an arbitrary constant.} \qquad (6.2.8)$$

Solution: By (6.1.19), $d\,w(t)/dt = \sum_{i=1}^{n} \det(\mathbf{D}_i)$, where

$$\mathbf{D}_i = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ w'_{i1} & w'_{i2} & \cdots & w'_{in} \\ \vdots & \vdots & \cdots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{pmatrix} = \mathbf{W} + \mathbf{e}_i\mathbf{e}_i^T\mathbf{W}' - \mathbf{e}_i\mathbf{e}_i^T\mathbf{W}.$$

Notice that $\left(-\mathbf{e}_i\mathbf{e}_i^T\mathbf{W}\right)$ subtracts $\mathbf{W}_{i*}$ from the i^{th} row while $\left(+\mathbf{e}_i\mathbf{e}_i^T\mathbf{W}'\right)$ adds $\mathbf{W}'_{i*}$ to the i^{th} row. Use the fact that $\mathbf{W}' = \mathbf{AW}$ to write

$$\mathbf{D}_i = \mathbf{W}+\mathbf{e}_i\mathbf{e}_i^T\mathbf{W}'-\mathbf{e}_i\mathbf{e}_i^T\mathbf{W} = \mathbf{W}+\mathbf{e}_i\mathbf{e}_i^T\mathbf{AW}-\mathbf{e}_i\mathbf{e}_i^T\mathbf{W} = \left(\mathbf{I}+\mathbf{e}_i\left(\mathbf{e}_i^T\mathbf{A}-\mathbf{e}_i^T\right)\right)\mathbf{W},$$

and apply formula (6.2.2) for the determinant of a rank-one updated matrix together with the product rule (6.1.15) to produce

$$\det(\mathbf{D}_i) = \left(1 + \mathbf{e}_i^T\mathbf{A}\mathbf{e}_i - \mathbf{e}_i^T\mathbf{e}_i\right)\det(\mathbf{W}) = a_{ii}(t)\,w(t),$$

so

$$\frac{d\,w(t)}{dt} = \sum_{i=1}^{n}\det(\mathbf{D}_i) = \left(\sum_{i=1}^{n} a_{ii}(t)\right) w(t) = \mathit{trace}\,\mathbf{A}(t)\,w(t).$$

In other words, $w(t)$ satisfies the first-order differential equation $w' = \tau\,w$, where $\tau = \mathit{trace}\,\mathbf{A}(t)$, and the solution of this equation is $w(t) = w(\xi_0)\,e^{\int_{\xi_0}^{t}\tau(\xi)\,d\xi}$.

Consequences: In addition to its aesthetic elegance, (6.2.8) is a useful result because it is the basis for the following theorems.

- If $\mathbf{x}' = \mathbf{Ax}$ has a set of solutions $\mathcal{S} = \{\mathbf{w}_1(t), \mathbf{w}_2(t), \ldots, \mathbf{w}_n(t)\}$ that is linearly independent at *some* point $\xi_0 \in (a,b)$, and if $\int_{\xi_0}^{t}\tau(\xi)\,d\xi$ is finite for $t \in (a,b)$, then $\mathcal{S}$ must be linearly independent at *every* point $t \in (a,b)$.
- If $\mathbf{A}$ is a constant matrix, and if $\mathcal{S}$ is a set of n solutions that is linearly independent at *some* value $t = \xi_0$, then $\mathcal{S}$ must be linearly independent for *all* values of t.

Proof. If $\mathcal{S}$ is linearly independent at ξ_0, then $\mathbf{W}(\xi_0)$ is nonsingular, so $w(\xi_0) \neq 0$. If $\int_{\xi_0}^{t}\tau(\xi)\,d\xi$ is finite when $t \in (a,b)$, then $e^{\int_{\xi_0}^{t}\tau(\xi)\,d\xi}$ is finite and nonzero on (a,b), so, by (6.2.8), $w(t) \neq 0$ on (a,b). Therefore, $\mathbf{W}(t)$ is nonsingular for $t \in (a,b)$, and thus $\mathcal{S}$ is linearly independent at each $t \in (a,b)$.

Exercises for section 6.2

6.2.1. Use a cofactor expansion to evaluate each of the following determinants.

(a) $\begin{vmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ -2 & 2 & 1 \end{vmatrix}$, (b) $\begin{vmatrix} 0 & 0 & -2 & 3 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 2 & 1 \\ 0 & 2 & -3 & 0 \end{vmatrix}$, (c) $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$.

6.2.2. Use determinants to compute the following inverses.

(a) $\begin{pmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ -2 & 2 & 1 \end{pmatrix}^{-1}$. (b) $\begin{pmatrix} 0 & 0 & -2 & 3 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 2 & 1 \\ 0 & 2 & -3 & 0 \end{pmatrix}^{-1}$.

6.2.3. (a) Use Cramer's rule to solve

$$\begin{aligned} x_1 + x_2 + x_3 &= 1, \\ x_1 + x_2 \quad\quad &= \alpha, \\ x_2 + x_3 &= \beta. \end{aligned}$$

(b) Evaluate $\lim_{t\to\infty} x_2(t)$, where $x_2(t)$ is defined by the system

$$\begin{aligned} x_1 + tx_2 + t^2x_3 &= t^4, \\ t^2x_1 + x_2 + tx_3 &= t^3, \\ tx_1 + t^2x_2 + x_3 &= 0. \end{aligned}$$

6.2.4. Is the following equation a valid derivation of Cramer's rule for solving a nonsingular system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}_i$ is as described on p. 476?

$$\frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})} = \det\left(\mathbf{A}^{-1}\mathbf{A}_i\right) = \det\left[\mathbf{e}_1 \cdots \mathbf{e}_{i-1}\ \mathbf{x}\ \mathbf{e}_{i+1} \cdots \mathbf{e}_n\right] = x_i.$$

6.2.5. (a) By example, show that $\det(\mathbf{A}+\mathbf{B}) \neq \det(\mathbf{A}) + \det(\mathbf{B})$.

(b) Using square matrices, construct an example that shows that

$$\det\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \neq \det(\mathbf{A})\det(\mathbf{D}) - \det(\mathbf{B})\det(\mathbf{C}).$$

6.2.6. Suppose $rank(\mathbf{B}_{m\times n}) = n$, and let $\mathbf{Q}$ be the orthogonal projector onto $N\left(\mathbf{B}^T\right)$. For $\mathbf{A} = [\mathbf{B}\,|\,\mathbf{c}_{n\times 1}]$, prove $\mathbf{c}^T\mathbf{Qc} = \det\left(\mathbf{A}^T\mathbf{A}\right)/\det\left(\mathbf{B}^T\mathbf{B}\right)$.

6.2.7. If $\mathbf{A}_{n\times n}$ is a nonsingular matrix, and if $\mathbf{D}$ and $\mathbf{C}$ are $n \times k$ matrices, explain how to use (6.2.1) to derive the formula

$$\det\left(\mathbf{A} + \mathbf{CD}^T\right) = \det(\mathbf{A})\det\left(\mathbf{I}_k + \mathbf{D}^T\mathbf{A}^{-1}\mathbf{C}\right).$$

Note: This is a generalization of (6.2.3) because if $\mathbf{c}_i$ and $\mathbf{d}_i$ are the i^{th} columns of $\mathbf{C}$ and $\mathbf{D}$, respectively, then

$$\mathbf{A} + \mathbf{CD}^T = \mathbf{A} + \mathbf{c}_1\mathbf{d}_1^T + \mathbf{c}_2\mathbf{d}_2^T + \cdots + \mathbf{c}_k\mathbf{d}_k^T.$$

6.2.8. Explain why $\mathbf{A}$ is singular if and only if $\mathbf{A}[\mathrm{adj}\,(\mathbf{A})] = \mathbf{0}$.

6.2.9. For a nonsingular linear system $\mathbf{Ax} = \mathbf{b}$, explain why each component of the solution must vary continuously with the entries of $\mathbf{A}$.

6.2.10. For scalars α, explain why $\mathrm{adj}\,(\alpha\mathbf{A}) = \alpha^{n-1}\mathrm{adj}\,(\mathbf{A})$. **Hint:** Recall Exercise 6.1.11.

6.2.11. For an $n \times n$ matrix $\mathbf{A}$, prove that the following statements are true.

(a) If $rank\,(\mathbf{A}) < n - 1$, then $\mathrm{adj}\,(\mathbf{A}) = \mathbf{0}$.

(b) If $rank\,(\mathbf{A}) = n - 1$, then $rank\,(\mathrm{adj}\,(\mathbf{A})) = 1$.

(c) If $rank\,(\mathbf{A}) = n$, then $rank\,(\mathrm{adj}\,(\mathbf{A})) = n$.

6.2.12. In 1812, Cauchy discovered the formula that says that if $\mathbf{A}$ is $n \times n$, then $\det\,(\mathrm{adj}\,(\mathbf{A})) = [\det\,(\mathbf{A})]^{n-1}$. Establish Cauchy's formula.

6.2.13. For the following tridiagonal matrix, $\mathbf{A}_n$, let $D_n = \det\,(\mathbf{A}_n)$, and derive the formula $D_n = 2D_{n-1} - D_{n-2}$ to deduce that $D_n = n + 1$.

$$\mathbf{A}_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}_{n\times n}.$$

6.2.14. By considering rank-one updated matrices, derive the following formulas.

(a)
$$\begin{vmatrix} \frac{1+\alpha_1}{\alpha_1} & 1 & \cdots & 1 \\ 1 & \frac{1+\alpha_2}{\alpha_2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \frac{1+\alpha_n}{\alpha_n} \end{vmatrix} = \frac{1 + \sum \alpha_i}{\prod \alpha_i}.$$

(b)
$$\begin{vmatrix} \alpha & \beta & \beta & \cdots & \beta \\ \beta & \alpha & \beta & \cdots & \beta \\ \beta & \beta & \alpha & \cdots & \beta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta & \beta & \beta & \cdots & \alpha \end{vmatrix}_{n\times n} = \begin{cases} (\alpha-\beta)^n \left(1 + \frac{n\beta}{\alpha-\beta}\right) & \text{if } \alpha \neq \beta, \\ 0 & \text{if } \alpha = \beta. \end{cases}$$

(c)
$$\begin{vmatrix} 1+\alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & 1+\alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \cdots & 1+\alpha_n \end{vmatrix} = 1 + \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

6.2.15. A ***bordered matrix*** has the form $\mathbf{B} = \begin{pmatrix} \mathbf{A} & \mathbf{x} \\ \mathbf{y}^T & \alpha \end{pmatrix}$ in which $\mathbf{A}_{n\times n}$ is nonsingular, $\mathbf{x}$ is a column, $\mathbf{y}^T$ is a row, and α is a scalar. Explain why the following statements must be true.

(a) $\begin{vmatrix} \mathbf{A} & \mathbf{x} \\ \mathbf{y}^T & -1 \end{vmatrix} = -\det\left(\mathbf{A} + \mathbf{x}\mathbf{y}^T\right).$ (b) $\begin{vmatrix} \mathbf{A} & \mathbf{x} \\ \mathbf{y}^T & 0 \end{vmatrix} = -\mathbf{y}^T \mathrm{adj}\,(\mathbf{A})\,\mathbf{x}.$

6.2.16. If $\mathbf{B}$ is $m \times n$ and $\mathbf{C}$ is $n \times m$, explain why (6.2.1) guarantees that $\lambda^m \det(\lambda\mathbf{I}_n - \mathbf{CB}) = \lambda^n \det(\lambda\mathbf{I}_m - \mathbf{BC})$ is true for all scalars λ.

6.2.17. For a square matrix $\mathbf{A}$ and column vectors $\mathbf{c}$ and $\mathbf{d}$, derive the following two extensions of formula (6.2.3).

(a) If $\mathbf{Ax} = \mathbf{c}$, then $\det\left(\mathbf{A} + \mathbf{c}\mathbf{d}^T\right) = \det(\mathbf{A})\left(1 + \mathbf{d}^T\mathbf{x}\right)$.

(b) If $\mathbf{y}^T\mathbf{A} = \mathbf{d}^T$, then $\det\left(\mathbf{A} + \mathbf{c}\mathbf{d}^T\right) = \det(\mathbf{A})\left(1 + \mathbf{y}^T\mathbf{c}\right)$.

6.2.18. Describe the determinant of an elementary reflector (p. 324) and a plane rotation (p. 333), and then explain how to find $\det(\mathbf{A})$ using Householder reduction (p. 341) and Givens reduction (Example 5.7.2).

6.2.19. Suppose that $\mathbf{A}$ is a nonsingular matrix whose entries are integers. Prove that the entries in $\mathbf{A}^{-1}$ are integers if and only if $\det(\mathbf{A}) = \pm 1$.

6.2.20. Let $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{v}^T$ be a matrix in which $\mathbf{u}$ and $\mathbf{v}$ are column vectors with integer entries.

(a) Prove that $\mathbf{A}^{-1}$ has integer entries if and only if $\mathbf{v}^T\mathbf{u} = 0$ or 1.

(b) A matrix is said to be ***involutory*** whenever $\mathbf{A}^{-1} = \mathbf{A}$. Explain why $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{v}^T$ is involutory when $\mathbf{v}^T\mathbf{u} = 1$.

6.2.21. Use induction to argue that a cofactor expansion of $\det(\mathbf{A}_{n\times n})$ requires

$$c(n) = n!\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!}\right)$$

multiplications for $n \geq 2$. Assume a computer will do 1,000,000 multiplications per second, and neglect all other operations to estimate how long it will take to evaluate the determinant of a 100×100 matrix using cofactor expansions. **Hint:** Recall the series expansion for e^x, and use $100! \approx 9.33 \times 10^{157}$.

6.2.22. Determine all values of λ for which the matrix $\mathbf{A}-\lambda\mathbf{I}$ is singular, where

$$\mathbf{A} = \begin{pmatrix} 0 & -3 & -2 \\ 2 & 5 & 2 \\ -2 & -3 & 0 \end{pmatrix}.$$

Hint: If $p(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$ is a monic polynomial with integer coefficients, then the integer roots of $p(\lambda)$ are a subset of the factors of α_0.

6.2.23. Suppose that $f_1(t),\ f_2(t), \ldots, f_n(t)$ are solutions of n^{th}-order linear differential equation $y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$, and let $w(t)$ be the Wronskian

$$w(t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix}.$$

By converting the n^{th}-order equation into a system of n first-order equations with the substitutions $x_1 = y,\ x_2 = y', \ldots, x_n = y^{(n-1)}$, show that $w(t) = w(\xi_0)\, \mathrm{e}^{-\int_{\xi_0}^{t} p_1(\xi)\, d\xi}$ for an arbitrary constant ξ_0.

6.2.24. Evaluate the ***Vandermonde determinant*** by showing

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{j>i}(x_j - x_i).$$

When is this nonzero (compare with Example 4.3.4)? **Hint:** For the polynomial $p(\lambda) = \begin{vmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-1} \end{vmatrix}_{k\times k}$, use induction to find the degree of $p(\lambda)$, the roots of $p(\lambda)$, and the coefficient of λ^{k-1} in $p(\lambda)$.

6.2.25. Suppose that each entry in $\mathbf{A}_{n\times n} = [a_{ij}(x)]$ is a differentiable function of a real variable x. Use formula (6.1.19) to derive the formula

$$\frac{d\big(\det(\mathbf{A})\big)}{dx} = \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{d\,a_{ij}}{dx}\, \mathring{A}_{ij}.$$

6.2.26. Consider the entries of $\mathbf{A}$ to be independent variables, and use formula (6.1.19) to derive the formula

$$\frac{\partial \det(\mathbf{A})}{\partial a_{ij}} = \mathring{A}_{ij}.$$

6.2.27. Laplace's Expansion. In 1772, the French mathematician Pierre-Simon Laplace (1749–1827) presented the following generalized version of the cofactor expansion. For an $n \times n$ matrix $\mathbf{A}$, let

$$\mathbf{A}(i_1 i_2 \cdots i_k \,|\, j_1 j_2 \cdots j_k) = \text{the } k \times k \text{ submatrix of } \mathbf{A} \text{ that lies on the intersection of rows } i_1, i_2, \ldots, i_k \text{ with columns } j_1, j_2, \ldots, j_k,$$

and let

$$M(i_1 i_2 \cdots i_k \,|\, j_1 j_2 \cdots j_k) = \text{the } n-k \times n-k \text{ minor determinant obtained by deleting rows } i_1, i_2, \ldots, i_k \text{ and columns } j_1, j_2, \ldots, j_k \text{ from } \mathbf{A}.$$

The ***cofactor*** of $\mathbf{A}(i_1 \cdots i_k \,|\, j_1 \cdots j_k)$ is defined to be the signed minor

$$\mathring{A}(i_1 \cdots i_k \,|\, j_1 \cdots j_k) = (-1)^{i_1+\cdots+i_k+j_1+\cdots+j_k} M(i_1 \cdots i_k \,|\, j_1 \cdots j_k).$$

This is consistent with the definition of cofactor given earlier because if $\mathbf{A}(i \,|\, j) = a_{ij}$, then $\mathring{A}(i \,|\, j) = (-1)^{i+j} M(i \,|\, j) = (-1)^{i+j} M_{ij} = \mathring{A}_{ij}$. For each fixed set of row indices $1 \le i_1 < \cdots < i_k \le n$,

$$\det(\mathbf{A}) = \sum_{1 \le j_1 < \cdots < j_k \le n} \det \mathbf{A}(i_1 \cdots i_k \,|\, j_1 \cdots j_k) \mathring{A}(i_1 \cdots i_k \,|\, j_1 \cdots j_k).$$

Similarly, for each fixed set of column indices $1 \le j_1 < \cdots < j_k \le n$,

$$\det(\mathbf{A}) = \sum_{1 \le i_1 < \cdots < i_k \le n} \det \mathbf{A}(i_1 \cdots i_k \,|\, j_1 \cdots j_k) \mathring{A}(i_1 \cdots i_k \,|\, j_1 \cdots j_k).$$

Each of these sums contains $\binom{n}{k}$ terms. Use Laplace's expansion to evaluate the determinant of

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & -2 & 3 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 2 & 1 \\ 0 & 2 & -3 & 0 \end{pmatrix}$$

in terms of the first and third rows.

(b) Using formula (6.1.19) produces

$$\frac{d\Big(\det(\mathbf{A})\Big)}{dx} = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 4-x & -5 \\ 1 & -1 & 3-x \end{vmatrix} + \begin{vmatrix} 2-x & 3 & 4 \\ 0 & -1 & 0 \\ 1 & -1 & 3-x \end{vmatrix} + \begin{vmatrix} 2-x & 3 & 4 \\ 0 & 4-x & -5 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= (-x^2+7x-7) + (-x^2+5x-2) + (-x^2+6x-8)$$
$$= -3x^2+18x-17.$$

6.1.19. No—almost any 2×2 example will show that this cannot hold in general.

6.1.20. It was argued in Example 4.3.6 that if there is at least one value of x for which the Wronski matrix

$$\mathbf{W}(x) = \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}$$

is nonsingular, then $\mathcal{S}$ is a linearly independent set. This is equivalent to saying that if $\mathcal{S}$ is a linearly *dependent* set, then the Wronski matrix $\mathbf{W}(x)$ is singular for all values of x. But (6.1.14) insures that a matrix is singular if and only if its determinant is zero, so, if $\mathcal{S}$ is linearly dependent, then the Wronskian $w(x)$ must vanish for every value of x. The converse of this statement is false (Exercise 4.3.14).

6.1.21. (a) $(n!)(n-1)$ (b) 11×11 (c) About 9.24×10^{153} sec $\approx 3\times 10^{146}$ years (d) About 3×10^{150} mult/sec. (Now this would truly be a "super computer.")

Solutions for exercises in section 6. 2

6.2.1. (a) 8 (b) 39 (c) -3

6.2.2. (a) $\mathbf{A}^{-1} = \dfrac{\operatorname{adj}(\mathbf{A})}{\det(\mathbf{A})} = \dfrac{1}{8}\begin{pmatrix} 0 & 1 & -1 \\ -8 & 4 & 4 \\ 16 & -6 & -2 \end{pmatrix}$

(b) $\mathbf{A}^{-1} = \dfrac{\operatorname{adj}(\mathbf{A})}{\det(\mathbf{A})} = \dfrac{1}{39}\begin{pmatrix} -12 & 25 & -14 & 7 \\ -9 & 9 & 9 & 15 \\ -6 & 6 & 6 & -3 \\ 9 & 4 & 4 & -2 \end{pmatrix}$

6.2.3. (a) $x_1 = 1-\beta, \quad x_2 = \alpha+\beta-1, \quad x_3 = 1-\alpha$

(b) Cramer's rule yields

$$x_2(t)=\frac{\begin{vmatrix}1 & t^4 & t^2\\ t^2 & t^3 & t\\ t & 0 & 1\end{vmatrix}}{\begin{vmatrix}1 & t & t^2\\ t^2 & 1 & t\\ t & t^2 & 1\end{vmatrix}}=\frac{t\begin{vmatrix}t^4 & t^2\\ t^3 & t\end{vmatrix}+\begin{vmatrix}1 & t^4\\ t^2 & t^3\end{vmatrix}}{\begin{vmatrix}1 & t\\ t^2 & 1\end{vmatrix}-t\begin{vmatrix}t^2 & t\\ t & 1\end{vmatrix}+t^2\begin{vmatrix}t^2 & 1\\ t & t^2\end{vmatrix}}$$

$$=\frac{t^3-t^6}{(t^3-1)(t^3-1)}=\frac{-t^3}{(t^3-1)},$$

and hence

$$\lim_{t\to\infty}x_2(t)=\lim_{t\to\infty}\frac{-1}{1-1/t^3}=-1.$$

6.2.4. Yes.

6.2.5. (a) Almost any two matrices will do the job. One example is $\mathbf{A}=\mathbf{I}$ and $\mathbf{B}=-\mathbf{I}$.

(b) Again, almost anything you write down will serve the purpose. One example is $\mathbf{A}=\mathbf{D}=\mathbf{0}_{2\times 2},\mathbf{B}=\mathbf{C}=\mathbf{I}_{2\times 2}$.

6.2.6. Recall from Example 5.13.3 that $\mathbf{Q}=\mathbf{I}-\mathbf{B}\mathbf{B}^T\mathbf{B}^{-1}\mathbf{B}^T$. According to (6.2.1),

$$\det\left(\mathbf{A}^T\mathbf{A}\right)=\det\begin{pmatrix}\mathbf{B}^T\mathbf{B} & \mathbf{B}^T\mathbf{c}\\ \mathbf{c}^T\mathbf{B} & \mathbf{c}^T\mathbf{c}\end{pmatrix}=\det\left(\mathbf{B}^T\mathbf{B}\right)\left(\mathbf{c}^T\mathbf{Q}\mathbf{c}\right).$$

Since $\det\left(\mathbf{B}^T\mathbf{B}\right)>0$ (by Exercise 6.1.10), $\mathbf{c}^T\mathbf{Q}\mathbf{c}=\det\left(\mathbf{A}^T\mathbf{A}\right)/\det\left(\mathbf{B}^T\mathbf{B}\right)$.

6.2.7. Expand $\begin{vmatrix}\mathbf{A} & -\mathbf{C}\\ \mathbf{D}^T & \mathbf{I}_k\end{vmatrix}$ both of the ways indicated in (6.2.1).

6.2.8. The result follows from Example 6.2.8, which says $\mathbf{A}[\mathrm{adj}\,(\mathbf{A})]=\det(\mathbf{A})\,\mathbf{I}$, together with the fact that $\mathbf{A}$ is singular if and only if $\det(\mathbf{A})=0$.

6.2.9. The solution is $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$, and Example 6.2.7 says that the entries in $\mathbf{A}^{-1}$ are continuous functions of the entries in $\mathbf{A}$. Since $x_i=\sum_k[\mathbf{A}^{-1}]_{ik}b_k$, and since the sum of continuous functions is again continuous, it follows that each x_i is a continuous function of the a_{ij} 's.

6.2.10. If $\mathbf{B}=\alpha\mathbf{A}$, then Exercise 6.1.11 implies $\mathring{B}_{ij}=\alpha^{n-1}\mathring{A}_{ij}$, so $\mathring{\mathbf{B}}=\alpha^{n-1}\mathring{\mathbf{A}}$, and hence $\mathrm{adj}\,(\mathbf{B})=\alpha^{n-1}\mathrm{adj}\,(\mathbf{A})$.

6.2.11. (a) We saw in §6.1 that $rank\,(\mathbf{A})$ is the order of the largest nonzero minor of $\mathbf{A}$. If $rank\,(\mathbf{A})<n-1$, then every minor of order $n-1$ (as well as $\det(\mathbf{A})$ itself) must be zero. Consequently, $\mathring{\mathbf{A}}=\mathbf{0}$, and thus $\mathrm{adj}\,(\mathbf{A})=\mathring{\mathbf{A}}^T=\mathbf{0}$.

(b)
$$\begin{aligned}rank\,(\mathbf{A})=n-1 &\implies \text{at least one minor of order } n-1 \text{ is nonzero}\\ &\implies \text{some } \mathring{A}_{ij}\neq 0 \implies \mathrm{adj}\,(\mathbf{A})\neq\mathbf{0}\\ &\implies rank\,(\mathrm{adj}\,(\mathbf{A}))\geq 1.\end{aligned}$$

$$\begin{aligned}
\text{Also,} \quad rank(\mathbf{A}) = n-1 &\implies \det(\mathbf{A}) = 0 \\
&\implies \mathbf{A}[\operatorname{adj}(\mathbf{A})] = \mathbf{0} \quad \text{(by Exercise 6.2.8)} \\
&\implies R(\operatorname{adj}(\mathbf{A})) \subseteq N(\mathbf{A}) \\
&\implies \dim R(\operatorname{adj}(\mathbf{A})) \leq \dim N(\mathbf{A}) \\
&\implies rank(\operatorname{adj}(\mathbf{A})) \leq n - rank(\mathbf{A}) = 1.
\end{aligned}$$

(c) $rank(\mathbf{A}) = n \implies \det(\mathbf{A}) \neq 0 \implies \operatorname{adj}(\mathbf{A}) = \det(\mathbf{A})\mathbf{A}^{-1}$
$\implies rank(\operatorname{adj}(\mathbf{A})) = n$

6.2.12. If $\det(\mathbf{A}) = 0$, then Exercise 6.2.11 insures that $rank(\operatorname{adj}(\mathbf{A})) \leq 1$. Consequently, $\det(\operatorname{adj}(\mathbf{A})) = 0$, and the result is trivially true because both sides are zero. If $\det(\mathbf{A}) \neq 0$, apply the product rule (6.1.15) to $\mathbf{A}[\operatorname{adj}(\mathbf{A})] = \det(\mathbf{A})\mathbf{I}$ (from Example 6.2.8) to obtain $\det(\mathbf{A})\det(\operatorname{adj}(\mathbf{A})) = [\det(\mathbf{A})]^n$, so that $\det(\operatorname{adj}(\mathbf{A})) = [\det(\mathbf{A})]^{n-1}$.

6.2.13. Expanding in terms of cofactors of the first row produces $D_n = 2\mathring{A}_{11} - \mathring{A}_{12}$. But $\mathring{A}_{11} = D_{n-1}$ and expansion using the first column yields

$$\mathring{A}_{12} = (-1)\begin{vmatrix} -1 & -1 & 0 & \cdots & 0 \\ 0 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{vmatrix} = (-1)(-1)D_{n-2},$$

so $D_n = 2D_{n-1} - D_{n-2}$. By recursion (or by direct substitution), it is easy to see that the solution of this equation is $D_n = n+1$.

6.2.14. (a) Use the results of Example 6.2.1 with $\lambda_i = 1/\alpha_i$.

(b) Recognize that the matrix $\mathbf{A}$ is a rank-one updated matrix in the sense that

$$\mathbf{A} = (\alpha - \beta)\mathbf{I} + \beta\mathbf{e}\mathbf{e}^T, \quad \text{where} \quad \mathbf{e} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

If $\alpha = \beta$, then $\mathbf{A}$ is singular, so $\det(\mathbf{A}) = 0$. If $\alpha \neq \beta$, then (6.2.3) may be applied to obtain

$$\det(\mathbf{A}) = \det\Big((\alpha-\beta)\mathbf{I}\Big)\left(1 + \frac{\beta\mathbf{e}^T\mathbf{e}}{\alpha-\beta}\right) = (\alpha-\beta)^n\left(1 + \frac{n\beta}{\alpha-\beta}\right).$$

(c) Recognize that the matrix is $\mathbf{I} + \mathbf{e}\mathbf{d}^T$, where

$$\mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Apply (6.2.2) to produce the desired formula.

6.2.15. (a) Use the second formula in (6.2.1).

(b) Apply the first formula in (6.2.1) along with (6.2.7).

6.2.16. If $\lambda = 0$, then the result is trivially true because both sides are zero. If $\lambda \neq 0$, then expand $\begin{vmatrix} \lambda\mathbf{I}_m & \lambda\mathbf{B} \\ \mathbf{C} & \lambda\mathbf{I}_n \end{vmatrix}$ both of the ways indicated in (6.2.1).

6.2.17. (a) Use the product rule (6.1.15) together with (6.2.2) to write

$$\mathbf{A} + \mathbf{c}\mathbf{d}^T = \mathbf{A} + \mathbf{A}\mathbf{x}\mathbf{d}^T = \mathbf{A}\left(\mathbf{I} + \mathbf{x}\mathbf{d}^T\right).$$

(b) Apply the same technique used in part (a) to obtain

$$\mathbf{A} + \mathbf{c}\mathbf{d}^T = \mathbf{A} + \mathbf{c}\mathbf{y}^T\mathbf{A} = \left(\mathbf{I} + \mathbf{c}\mathbf{y}^T\right)\mathbf{A}.$$

6.2.18. For an elementary reflector $\mathbf{R} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T/\mathbf{u}^T\mathbf{u}$, (6.2.2) insures $\det(\mathbf{R}) = -1$. If $\mathbf{A}_{n\times n}$ is reduced to upper-triangular form (say $\mathbf{PA} = \mathbf{T}$) by Householder reduction as explained on p. 341, then $\det(\mathbf{P})\det(\mathbf{A}) = \det(\mathbf{T}) = t_{11}\cdots t_{nn}$. Since $\mathbf{P}$ is the product of elementary reflectors, $\det(\mathbf{A}) = (-1)^k t_{11}\cdots t_{nn}$, where k is the number of reflections used in the reduction process. In general, one reflection is required to annihilate entries below a diagonal position, so, if no reduction steps can be skipped, then $\det(\mathbf{A}) = (-1)^{n-1} t_{11}\cdots t_{nn}$. If $\mathbf{P}_{ij}$ is a plane rotation, then there is a permutation matrix (a product of interchange matrices) $\mathbf{B}$ such that $\mathbf{P}_{ij} = \mathbf{B}^T\begin{pmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}\mathbf{B}$, where $\mathbf{Q} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$ with $c^2 + s^2 = 1$. Consequently, $\det(\mathbf{P}_{ij}) = \det\left(\mathbf{B}^T\right)\begin{vmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{vmatrix}\det(\mathbf{B}) = \det(\mathbf{Q}) = 1$ because $\det(\mathbf{B})\det\left(\mathbf{B}^T\right) = \det(\mathbf{B})^2 = 1$ by (6.1.9). Since Givens reduction produces $\mathbf{PA} = \mathbf{T}$, where $\mathbf{P}$ is a product of plane rotations and $\mathbf{T}$ is upper triangular, the product rule (6.1.15) insures $\det(\mathbf{P}) = 1$, so $\det(\mathbf{A}) = \det(\mathbf{T}) = t_{11}\cdots t_{nn}$.

6.2.19. If $\det(\mathbf{A}) = \pm 1$, then (6.2.7) implies $\mathbf{A}^{-1} = \pm\mathrm{adj}(\mathbf{A})$, and thus $\mathbf{A}^{-1}$ is an integer matrix because the cofactors are integers. Conversely, if $\mathbf{A}^{-1}$ is an integer matrix, then $\det\left(\mathbf{A}^{-1}\right)$ and $\det(\mathbf{A})$ are both integers. Since

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \implies \det(\mathbf{A})\det\left(\mathbf{A}^{-1}\right) = 1,$$

it follows that $\det(\mathbf{A}) = \pm 1$.

6.2.20. (a) Exercise 6.2.19 guarantees that $\mathbf{A}^{-1}$ has integer entries if and only if $\det(\mathbf{A}) = \pm 1$, and (6.2.2) says that $\det(\mathbf{A}) = 1 - 2\mathbf{v}^T\mathbf{u}$, so $\mathbf{A}^{-1}$ has integer entries if and only if $\mathbf{v}^T\mathbf{u}$ is either 0 or 1.

(b) According to (3.9.1),

$$\mathbf{A}^{-1} = \left(\mathbf{I} - 2\mathbf{u}\mathbf{v}^T\right)^{-1} = \mathbf{I} - \frac{2\mathbf{u}\mathbf{v}^T}{2\mathbf{v}^T\mathbf{u} - 1},$$

and thus $\mathbf{A}^{-1} = \mathbf{A}$ when $\mathbf{v}^T\mathbf{u} = 1$.

6.2.21. For $n = 2$, two multiplications are required, and $c(2) = 2$. Assume $c(k)$ multiplications are required to evaluate any $k \times k$ determinant by cofactors. For a $k+1 \times k+1$ matrix, the cofactor expansion in terms of the i^{th} row is

$$\det(\mathbf{A}) = a_{i1}\mathring{A}_{i1} + \cdots + a_{ik}\mathring{A}_{ik} + a_{ik+1}\mathring{A}_{ik+1}.$$

Each $\mathring{A}_{ij}$ requires $c(k)$ multiplications, so the above expansion contains

$$\begin{aligned}(k+1) + (k+1)c(k) &= (k+1) + (k+1)k!\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(k-1)!}\right)\\ &= (k+1)!\left(\frac{1}{k!} + \left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(k-1)!}\right)\right)\\ &= c(k+1)\end{aligned}$$

multiplications. Remember that $e^x = 1+x+x^2/2!+x^3/3!+\cdots$, so for $n = 100$,

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{99!} \approx e - 1,$$

and $c(100) \approx 100!(e-1)$. Consequently, approximately 1.6×10^{152} seconds (i.e., 5.1×10^{144} years) are required.

6.2.22. $\mathbf{A} - \lambda\mathbf{I}$ is singular if and only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. The cofactor expansion in terms of the first row yields

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= -\lambda\begin{vmatrix} 5-\lambda & 2\\ -3 & -\lambda\end{vmatrix} + 3\begin{vmatrix} 2 & 2\\ -2 & -\lambda\end{vmatrix} - 2\begin{vmatrix} 2 & 5-\lambda\\ -2 & -3\end{vmatrix}\\ &= -\lambda^3 + 5\lambda^2 - 8\lambda + 4,\end{aligned}$$

so $\mathbf{A} - \lambda\mathbf{I}$ is singular if and only if $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$. According to the hint, the integer roots of $p(\lambda) = \lambda^3 - 5\lambda^2 + 8\lambda - 4$ are a subset of $\{\pm4, \pm2, \pm1\}$. Evaluating $p(\lambda)$ at these points reveals that $\lambda = 2$ is a root, and either ordinary or synthetic division produces

$$\frac{p(\lambda)}{\lambda - 2} = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1).$$

Therefore, $p(\lambda) = (\lambda - 2)^2(\lambda - 1)$, so $\lambda = 2$ and $\lambda = 1$ are the roots of $p(\lambda)$, and these are the values for which $\mathbf{A} - \lambda\mathbf{I}$ is singular.

6.2.23. The indicated substitutions produce the system

$$\begin{pmatrix} x_1'\\ x_2'\\ \vdots\\ x_{n-1}'\\ x_n'\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -p_n & -p_{n-1} & -p_{n-2} & \cdots & -p_1\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_{n-1}\\ x_n\end{pmatrix}.$$

Each of the n vectors $\mathbf{w}_i = \begin{pmatrix} f_i(t) & f_i'(t) & \cdots & f_i^{(n-1)} \end{pmatrix}^T$ for $i = 1, 2, \ldots, n$ satisfies this system, so (6.2.8) may be applied to produce the desired conclusion.

6.2.24. The result is clearly true for $n = 2$. Assume the formula holds for $n = k - 1$, and prove that it must also hold for $n = k$. According to the cofactor expansion in terms of the first row, $\deg p(\lambda) = k - 1$, and it's clear that

$$p(x_2) = p(x_3) = \cdots = p(x_k) = 0,$$

so $x_2, x_3, \ldots, x_k$ are the $k - 1$ roots of $p(\lambda)$. Consequently,

$$p(\lambda) = \alpha(\lambda - x_2)(\lambda - x_3) \cdots (\lambda - x_k),$$

where α is the coefficient of λ^{k-1}. But the coefficient of λ^{k-1} is the cofactor associated with the $(1, k)$-entry, so the induction hypothesis yields

$$\alpha = (-1)^{k-1} \begin{vmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{k-2} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{k-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-2} \end{vmatrix}_{k-1 \times k-1} = (-1)^{k-1} \prod_{j > i \geq 2} (x_j - x_i).$$

Therefore,

$$\begin{aligned} \det(\mathbf{V}_k) = p(x_1) &= (x_1 - x_2)(x_1 - x_3) \cdots (x_1 - x_k)\alpha \\ &= (x_1 - x_2)(x_1 - x_3) \cdots (x_1 - x_k)\Big((-1)^{k-1} \prod_{j > i \geq 2} (x_j - x_i)\Big) \\ &= (x_2 - x_1)(x_3 - x_1) \cdots (x_k - x_1) \prod_{j > i \geq 2} (x_j - x_i) \\ &= \prod_{j > i} (x_j - x_i), \end{aligned}$$

and the formula is proven. The determinant is nonzero if and only if the x_i 's are distinct numbers, and this agrees with the conclusion in Example 4.3.4.

6.2.25. According to (6.1.19),

$$\frac{d\Big(\det(\mathbf{A})\Big)}{dx} = \det(\mathbf{D}_1) + \det(\mathbf{D}_2) + \cdots + \det(\mathbf{D}_n),$$

where $\mathbf{D}_i$ is the matrix

$$\mathbf{D}_i = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a'_{i1} & a'_{i2} & \cdots & a'_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Expanding $\det(\mathbf{D}_i)$ in terms of cofactors of the i^{th} row yields

$$\det(\mathbf{A}_i) = a'_{i1}\mathring{A}_{i1} + a'_{i2}\mathring{A}_{i2} + \cdots + a'_{in}\mathring{A}_{in},$$

so the desired conclusion is obtained.

6.2.26. According to (6.1.19),

$$\frac{\partial \det(\mathbf{A})}{\partial a_{ij}} = \det(\mathbf{D}_i) = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} \leftarrow \text{row } i = \mathring{A}_{ij}.$$

6.2.27. The $\binom{4}{2} = 6$ ways to choose pairs of column indices are

$$\begin{matrix} (1,2) & (1,3) & (1,4) \\ & (2,3) & (2,4) \\ & & (3,4) \end{matrix}$$

so that the Laplace expansion using $i_1 = 1$ and $i_2 = 3$ is

$$\begin{aligned} \det(\mathbf{A}) = {} & \det \mathbf{A}(1,3\,|\,1,2)\,\mathring{A}(1,3\,|\,1,2) + \det \mathbf{A}(1,3\,|\,1,3)\,\mathring{A}(1,3\,|\,1,3) \\ & + \det \mathbf{A}(1,3\,|\,1,4)\,\mathring{A}(1,3\,|\,1,4) + \det \mathbf{A}(1,3\,|\,2,3)\,\mathring{A}(1,3\,|\,2,3) \\ & + \det \mathbf{A}(1,3\,|\,2,4)\,\mathring{A}(1,3\,|\,2,4) + \det \mathbf{A}(1,3\,|\,3,4)\,\mathring{A}(1,3\,|\,3,4) \\ = {} & 0 + (-2)(-4) + (-1)(3)(-2) + 0 + (-3)(-3) + (-1)(-8)(2) \\ = {} & 39. \end{aligned}$$