The expected frequency-of-scores set for a $(n,r)$ maze is therefore:

$$\left[ {}_{\nu}T^{(\alpha)}(r,n) \Big/ \binom{x}{r} \right], \ \alpha = 0, 1, 2 \ldots n.$$

These and other statistics concerning the distribution of scores may be used to define parameters for the maze which may be related to subjective difficulty.

In this notation $m_{ij}$ equals the number of pathways between dot $i$ and dot $j$ with zero score ($i, j = 0, 1, 2 \ldots r, f$): and if $m_{ij} > 0$, then we may say that dot $j$ is directly accessible from dot $i$. Developing this concept we may use a binary notation in the form of matrix $M' = (m'_{ij})$ ($i, j = 0, 1, 2 \ldots r, f$) where $m'_{ij} = 1$ if $m_{ij} > 0$ (that is, if $j$ is directly accessible from $i$) and $= 0$ otherwise. Matrices $V^{(a)\prime}$, $T^{(1)\prime}$, $T^{(2)\prime}$, $T^{(3)\prime}$ may also be derived corresponding to the unprimed matrices given here. The basic difference between the two sets of matrices is that in the first set the number of different pathways with score $\alpha$ between dots $i$ and $j$ is considered, whereas in the second series the number of different sets of $\alpha$ dots lying on pathways between dots $i$ and $j$ is the underlying concept. Thus, for example, ${}_{\nu}{}_{of}{}^{(m)\prime}$ gives the number of different solution sets of dots.

Further, the dots (not including $o$ and $f$) may be divided into $n$ groups corresponding to the $n$ horizontal maze rows, and the vector $(d_p)$ may be defined, where $d_p$ = number of dots on the $p^{\text{th}}$ horizontal maze row ($p = 1, 2 \ldots n$).

The matrix $M'$ may now be partitioned into sub-matrices $R_{a\beta}$ where $a, \beta$ refer to maze rows; here 'maze rows' are taken to include a zero row and an $f$ row and so $\alpha, \beta = 0, 1, 2 \ldots n, f$. $R_{a\beta}$ is a $d_a \times d_\beta$ matrix which gives direct accessibility relationships between dots on the $\alpha^{\text{th}}$ row and the $\beta^{\text{th}}$ row. Again, $R_{a\beta} = O$ if $\alpha \geqslant \beta$.

Since one theory describing maze problem-solving activity postulates that individuals differ in the size of the perceptual unit which they use, it seems relevant to illustrate one way in which the present approach could be used to analyse this aspect of the problem. To do this we have defined as a $\Delta$ any maze-linked set of dots. A $\Delta$ is designated by $\delta(\nu,\gamma)$ where $\nu$ is the number of dots in the $\Delta$ and $\gamma$ is the number of gaps, or vacant sites, interspersed between the dots. Now considering $M'$ in the partitioned form $R_{a\beta}$ ($a, \beta = 0, 1, 2 \ldots n, f$) which is an upper triangular matrix with zero sub-matrices $R_{aa}$ in the main diagonal, then it will be seen that the $\delta(\nu,\gamma)$'s are given by the $(\nu - 1 + \gamma)$th parallel of sub-matrices of $M'^{\nu-1}$. If the partitioned matrix $M'_{(\rho)}$ is defined so that $M'_{(\rho)}$ has the same first $\rho$ parallels of sub-matrices as $M'$ but all other parallels consist of zero sub-matrices, then it will be seen that the $\delta(\nu,\gamma)$'s are given by the $(\nu - 1 + \gamma)$th parallel of sub-matrices of $M'^{\nu-1}_{(\gamma+1)}$. In the particular case of the complete solution $\Delta$'s, that is, the $\delta(m + 2, n - m)$'s these are given by the $(n + 1)$th parallel of sub-matrices of $M'^{m+1}_{(n-m+1)}$. In other words, $M'_{(n-m+1)}$ and not $M'$ need be considered for the solution. Thus for a maze with $n = 16$, $m = 12$ (that is, $n - m + 1 = 5$), the subject need only consider direct accessibility relationships between dots which are separated by not more than five maze rows.

The foregoing mathematical analysis is one of several which might be equally or more valuable. It is presented in this form because it deals primarily with the target dot relationships and so lends itself to a study of the problem-solving activity involved. It has, we hope, been developed sufficiently to show that a systematic analysis of this test material is both practical and potentially fruitful.

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# DISTORTION OF VISUAL SPACE AS INAPPROPRIATE CONSTANCY SCALING

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DISTORTIONS of visual space associated with certain simple patterns have been investigated since the beginning of experimental psychology[1], and many theories have been proposed[2], but so far none, in my opinion, has been satisfactory in explaining these so-called 'geometrical' illusions. Figs. 1, 2 and 3 show representative illusions of the kind we are considering.

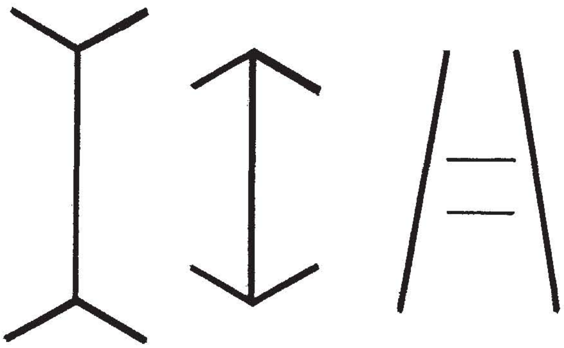
The traditional theories fall into three classes: (1) That certain shapes produce, or tend to produce, abnormal eye movements. (2) That some kind of central 'confusion' is produced by certain shapes, particularly non-parallel lines and corners. (3) That the figures suggest depth by perspective, and that this 'suggestion' in some way distorts visual space.

The eye movement theories are difficult to support because the illusions occur undiminished when the retinal image is optically stabilized on the retina[3], or when the figures are viewed as after-images following illumination by a bright flash of light. Further, since distortions can occur in opposed directions at the same time (as with the Müller–Lyer figure[4] (Fig. 1*a*) it is difficult to see how either overt or incipient eye movements could be involved. The various 'confusion' theories all suffer from vagueness, and they give us no idea as to why the distortions should occur in the observed directions, or only in certain kinds of figures. The perspective theory[2] is inadequate because it does not suggest why or how perspective should produce distortions in flat figures, but it does imply a generalization which seems to hold true of all the known illusion figures, and this gives a clue vital to understanding the origin of the illusions.

The illusion figures may be thought of as flat projections of typical views of objects lying in three-dimensional space. For example, the outward-going Müller–Lyer arrow figure is a typical projection of, say, the corner of a room—the fins representing the intersections of the walls with the ceiling and floor—while the in-going arrow is a typical projection of an outside corner of a house or a box, the converging lines receding into the distance. The following generalization seems to hold for all the illusion figures thought of in this way: The parts of the figures corresponding to distant objects are expanded and the parts corresponding to nearer objects are reduced. Thus in the Müller–Lyer figure the vertical line would be further away in the diverging case, and is expanded in the illusion, and vice versa, while in the Ponzo figure the upper horizontal line would be farther away and it also is expanded in the flat illusion figure.

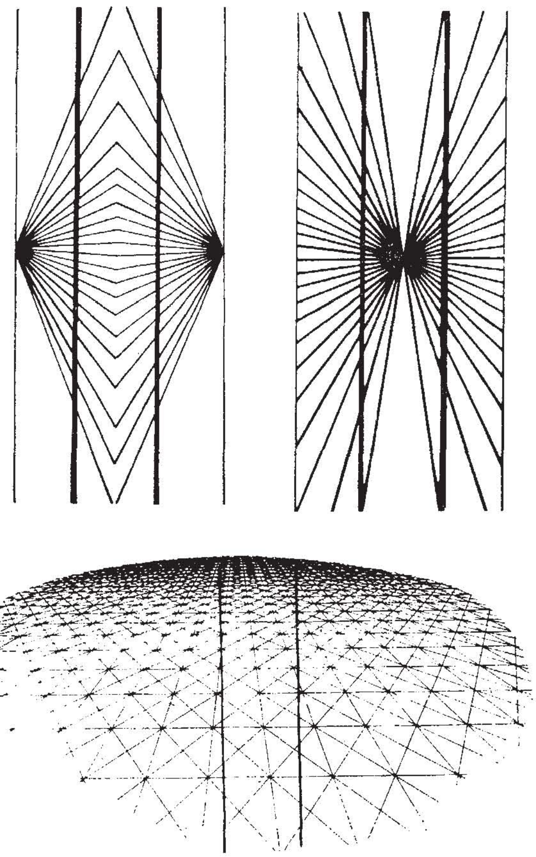
Given that this generalization holds for all the illusions, why should these distortions occur ?

Do we know of any other perceptual phenomena involving systematic perceptual modification of the retinal image ? There is a well-known set of phenomena which certainly does involve perceptual modification of retinal images—size constancy[5,6]. This is the tendency for objects to appear much the same size over a wide range of distance



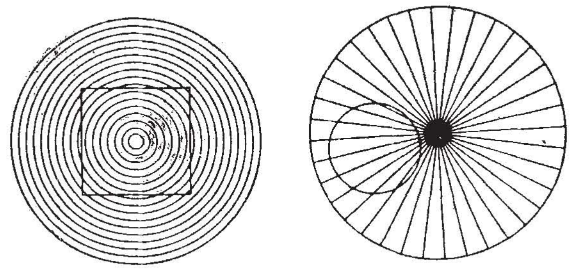
a b

Fig. 1. (*a*) The Müller-Lyer; (*b*) the Ponzo illusion



a b c

Fig. 2. (*a*) and (*b*) Alternative forms of the Hering illusion. The vertical lines are bowed inwards and outwards, respectively. (*c*) An illusion showing how parallel lines indicating distance seem to diverge when presented on a texture gradient. (The texture taken from Gibson, *The Perception of the Visual World*, 1951)



a b

Fig. 3. Further distortions to be expected on the distance hypothesis; the concentric circles and spokes set the constancy scaling by indicating depth. (Figures, though not interpretation, from Orbison, *Amer. J. Psychol.*, **52**, 39; 1939)

in spite of the changes of the retinal images associated with distance of the object. We may refer to the processes involved as constancy scaling. Now in constancy scaling we find known processes which not only could but also must produce distortion of visual space if the scaling were set inappropriately to the distance of an observed object. It is strange that apparently only one writer, Tausch, has considered constancy in connexion with the geometrical illusions[7].

We can see our own scaling system at work in the following demonstration of Emmert's law[8]. The after-image of a bright light is 'projected' on to a series of screens lying at various distances, or a single screen moved away or towards the observer. Although the effective retinal image is constant, the after-image perceived as lying on a screen looks larger the farther the screen is from the observer. Complete constancy would give a doubling in size for each doubling of distance, and the amount of scaling can be quantified under various conditions for stationary or moving screens[9,10].

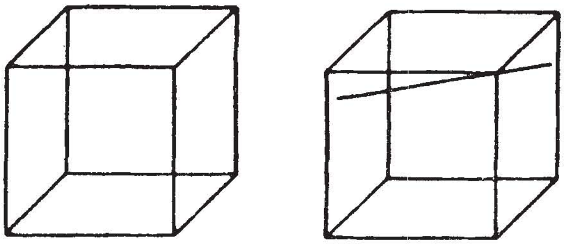
Clearly inappropriate constancy scaling would produce distortion of visual space, but why should this occur with the illusion figures which are in fact flat and are generally seen to be flat ? It is generally assumed that constancy scaling depends simply on apparent distance (as Emmert's law might suggest); but if we are to suppose that constancy scaling can operate for figures clearly lying on a flat surface we must challenge this assumption, and suggest that visual features associated with distance can modify constancy scaling even when no depth is seen. If we are to suppose that the illusions are due to misplaced constancy scaling, we must suppose that the scaling can be set directly by depth features of flat figures, and that the scaling is not set simply as a function of apparent distance as is generally thought to be the case.

Perspective drawings and photographs are seen to depict objects as if they lay in three dimensions, and yet at the same time they appear flat, lying on the plane of the paper, and so they are perceptually paradoxical. The surface texture of the paper evidently prevents the perspective from making the objects appear truly three dimensional, for if we remove all texture and view with one eye, then perspective drawings can look as impressively in depth as the real world viewed with one eye.

We have presented the well-known illusion figures with no background texture—by making wire models coated in luminous paint so that they glow in the dark, or using back illuminated transparencies—and we find that, viewed with one eye, they look three dimensional, provided the angles are not marked exaggerations of perspective. The Müller–Lyer arrows, for example, look like corners and not like flat projections when presented as luminous figures in the dark, and those parts which appear most distant are the parts which are expanded in the illusions as normally presented on textured paper. What happens to the distortions when we remove the background texture is complex, and will be discussed more fully elsewhere; but, in general, distortions are reduced or disappear.

Emmert's law may suggest that constancy scaling arises directly from apparent distance; but there is retinal information indicating the distance of each position of the screen, and possibly this might serve directly to set the scaling. However, the following demonstration shows conclusively that scaling can occur simply as a function of apparent depth and independently of retinal or other sensory information.

Fig. 4*a* shows the well-known Necker cube figure—a skeleton cube which reverses spontaneously in depth so that sometimes one face, sometimes another, appears the nearer. As shown on textured paper, it is paradoxical in the manner described here—it looks as if it were in depth and yet it is seen to be flat on the paper. By making a luminous model of this figure, and viewing it in the dark, we find that it still reverses but now it looks like a true three-dimensional figure, and it undergoes size changes—the apparently farther face looking somewhat larger than the nearer, showing that constancy scaling is now operating. Since the retinal image remains unchanged it follows that the scaling is set under these conditions as a simple function of apparent distance. This is shown most dramatically with a three-dimensional luminous cube. This looks like a true cube when seen correctly, but when perceptually reversed in depth it looks like a truncated pyramid, the apparently front face being the smaller[11].



*a* *b*

Fig. 4. (*a*) The Necker cube. This reverses in depth. When viewed as a self-luminous figure, the apparent front looks smaller, the back larger. (*b*) Humphrey's figure. The oblique line is seen as slightly bent; the direction of bending being determined by the angle against which it is placed, and not by the way the cube appears to lie in depth

It thus appears that there are two ways in which constancy scaling can be set. We may name these:

(1) *Primary constancy scaling*. This is set by perspective or other features normally associated with distance. These features can be at variance with apparent distance in special cases, such as the illusion figures. (We call it 'primary' because it seems to be primitive, and to be mediated by neural systems situated early in the perceptual system.)

(2) *Secondary constancy scaling* is set simply by apparent distance, and this may be a function of previous knowledge and is not necessarily tied directly to visual information. Its existence is suggested but not proved by Emmert's law; but it is conclusively demonstrated with the ambiguous self-luminous objects which change their shape systematically according to which faces appear nearer or farther though there is no change in the retinal image. Errors in apparent distance should produce distortion of visual space via this secondary scaling system, and the well-known moon illusion may be an example.

Although the self-luminous figures do clearly demonstrate what we have called the secondary constancy scaling system, what clear evidence have we for the primary system, supposed to be set by typical depth cues even in the absence of depth perception ? For our present purpose it is much more important to demonstrate the existence of primary than secondary scaling. To get evidence for primary scaling entirely independent of the illusions is very difficult, but the following is at least suggestive.

(1) It has been noticed by Humphrey[12] that a straight line drawn across a corner of a Necker cube (Fig. 4*b*) appears bent. Now this is particularly interesting because the direction of bending is the same which ever way the cube appears to lie in depth. It is bent in the direction to be expected if constancy scaling is operating from the typical perspective interpretation of the angle against which the line lies.

(2) In primitive races living in houses without corners the geometrical illusions are reduced[13,14]. If learning is important, this would be expected.

(3) In a case of a man blind from the first few months of life, but gaining his sight after operation fifty years later, we have found that the illusions were largely absent, and his constancy appeared abnormal or absent although he could at that time, some weeks after the corneal graft operation, recognize common objects[15]. This has been noted in other cases. (In fact, it was this observation which suggested to me this kind of theory of the illusions.)

We should expect the different scaling systems to have somewhat different time-constants, and we are attempting to measure these to establish their separate existence quite apart from considerations of distortions of visual space.

It further may be suggested that figural after-effects—distortions similar to the geometrical illusions, but produced as a result of prolonged viewing of a suitable stimulus pattern and transferring to a second test pattern—may be due to the primary scaling being set by depth features present in the stimulus pattern, this scaling taking some time after lengthy fixation to become appropriate to the second test pattern, so the second pattern is distorted by scaling carried over from the earlier pattern. Preliminary experiments are providing strong evidence that figural after-effects can be thought of in this way, and such a theory would have advantages over present theories of the figural after-effects which are *ad hoc*, involve dubious physiological speculation and fail to make useful predictions[16,17].

In attempting to give a general account of all illusions involving systematic distortions of visual space, either while viewing a figure or following on prolonged viewing, and relating the distortions to a known perceptual phenomenon—size constancy—we have not attempted to specify the neural processes involved, and we believe this to be impossible at this time. Recent work on recording from the visual regions of the cat's brain while presenting the eyes with moving or fixed patterns[18] gives promise that the underlying neural mechanisms may soon be revealed.

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